Lecture #12

- Finished the proof of Theorem 1 from Lecture 11.
- For the rest of today, we shall focus on the center of $U_q(sl_2)$.

$$\boxed{\text{Theorem 1: If } q \neq \sqrt[4]{1}\text{, then the center of } U_q(sl_2) \text{ is generated by } C = FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}}$$

▶ Let $Z_q :=$ center of $U_q(sl_2)$.

- First, we note that the algebra $U_q(sl_2)$ is $\mathbb{Z}$-graded via ← check!

$$\boxed{\deg(E) = 1, \quad \deg(K^{\pm 1}) = 0, \quad \deg(F) = -1.}$$

So, we can write $U_q(sl_2) = \bigoplus_{n\in\mathbb{Z}} \underbrace{U_q(sl_2)_n}_{\text{degree } n \text{ component}}$

As $KE = q^2 EK$, $KF = q^{-2} FK$, we deduce that $Kx = q^{2n} xK \;\; \forall x \in U_q(sl_2)_n$. Thus, assuming $q \neq \sqrt[4]{1}$, we get $Z_q \subseteq U_q(sl_2)_0$.

- Evoking the PBW theorem and the above inclusion $Z_q \subseteq U_q(sl_2)_0$, we conclude that any central elt $z \in Z_q$ can be uniquely written as:

$$\boxed{z = \sum_{i\geq 0} F^i \cdot P_i \cdot E^i \quad \text{with } P_i \in \Bbbk[K, K^{-1}]}$$

We consider the following projection on U_q^0 (Harish-Chandra homomorphism):

$$\boxed{\pi : U_q(sl_2)_0 \longrightarrow U_q^0 \quad \text{given by} \quad \sum_{i\geq 0} F^i \cdot P_i \cdot E^i \longmapsto P_0}$$

Easy exercise: Verify that π is indeed a homomorphism

- The first relevance of π is due to the following simple observation:

> **Lemma 1**: If $z \in Z_q$ and V is a highest weight $U_q(sl_2)$-module with highest weight λ, then $zv = \underline{\pi(z)(\lambda)} \cdot v \;\; \forall v \in V$

here, we evaluate $\pi(z) \in \Bbbk[K^{\pm 1}]$ at $K \mapsto \lambda$

▶ Let $u \in V$ be the highest weight vector. Then $E^i u = 0 \;\; \forall i > 0$, $K^s u = \lambda^s u \;\; \forall s \in \mathbb{Z}$. Thus, $zu = \pi(z)u = \pi(z)(\lambda) \cdot u$. On the other hand, as V is generated by u, and $z \in Z_q$, we conclude $zv = \pi(z)(\lambda) \cdot v \;\; \forall v \in V$. ◼

- Next, we show that restriction of π on Z_q is injective:

> **Lemma 2**: If $z \in Z_q$ and $\pi(z) = 0$, then $z = 0$

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▸ Proof of Lemma 2

Let $\underset{\neq 0}{z} \in Z_q$ be s.t. $\pi(z)=0$. Then $z=\sum_{i=k}^{\ell} F^i \cdot P_i \cdot E^i$ for some $0<k\leq \ell$ with $P_k \neq 0$ & $P_\ell \neq 0$ (here, all $P_i \in \Bbbk[K, K^{-1}]$).

Key Idea: Consider a Verma module $M(\lambda)$ with $\underbrace{\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}}_{\text{which guarantees } M(\lambda)\text{-simple.}}$

Recall that $M(\lambda)$ has a basis $\{v_0, v_1, v_2, \ldots\}$. Let us compute $z(v_k)$.

Know: $\begin{cases} E^{>k} v_k = 0 \\ E^k v_k = c \cdot v_0 \text{ and } c\neq 0 \text{ for } \lambda \notin \pm q^{\mathbb{Z}_{\geq 0}} \end{cases} \Rightarrow z v_k = c \cdot F^k \underbrace{P_k v_0}_{=P_k(\lambda)\cdot v_0}$

Thus: $z v_k = c \cdot P_k(\lambda) \cdot v_k$

On the other hand, by Lemma 1, we have $z v_k = 0$ $\Big\} \overset{c\neq 0}{\Longrightarrow} P_k(\lambda) = 0 \quad \forall \lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$.

Hence, the Laurent pol-l $P_k \in \Bbbk[K, K^{-1}]$ is ZERO, contradiction. ∎

• Next, appealing again to Verma modules, we shall prove certain symmetry of $\pi(z)$. To this end, given any Laurent polynomial $P(K) \in \Bbbk[K, K^{-1}]$ we define

$$\boxed{\widetilde{P}(K) := P(q^{-1}K) \in \Bbbk[K, K^{-1}]}$$

$\boxed{\textbf{Lemma 3: } \text{For any } z \in Z_q, \text{ we have } \widetilde{\pi(z)}(\lambda) = \widetilde{\pi(z)}(\lambda^{-1}) \quad \forall \lambda \in \Bbbk^*}$

▸ Recall that $\forall n \in \mathbb{Z}_{\geq 0}$, we realized the irreducible $U_q(\mathfrak{sl}_2)$-module $L(n,+)$ as a quotient $M(q^n)/M(q^{-n-2})$. Here, we have a $U_q(\mathfrak{sl}_2)$-module embedding

$$\boxed{M(q^{-n-2}) \hookrightarrow M(q^n)}$$

Applying Lemma 1 and $z \in Z_q$, we get $\pi(z)(q^n) = \pi(z)(q^{-n-2})$ which is equivalent to $\widetilde{\pi(z)}(q^{n+1}) = \widetilde{\pi(z)}(q^{-(n+1)})$ $\forall n \in \mathbb{Z}_{\geq 0}$. It remains to note that a Laurent pol-l $p(K)$ satisfies $p(q^{n+1}) = p(q^{-(n+1)})$ $\forall n \in \mathbb{Z}_{\geq 0}$ if and only if $p(\lambda) = p(\lambda^{-1})$ $\forall \lambda \in \Bbbk^*$ (again, we use $q \neq \sqrt[?]{1}$). This completes the proof ∎

• The following is a simple exercise:

> **Lemma 4** (exercise): A Laurent pol-l $p(K) \in \Bbbk[K, K^{-1}]$ satisfies $p(\lambda) = p(\lambda^{-1})$ $\forall \lambda \in \Bbbk^*$ if and only if p is a polynomial in $K + K^{-1}$.

Now we are ready to finish the proof of Theorem 1.

On one hand, combining Lemmas 2-4, we find:

$$\boxed{\pi: Z_q \hookrightarrow \mathbb{k}[K,K^{-1}] \quad \text{with } \mathrm{Im}(\pi) \subseteq \mathbb{k}[qK+q^{-1}K^{-1}]}$$

On the other hand, we know $Z_q \ni C$ and clearly $\pi(C) = (qK+q^{-1}K^{-1})/(q-q^{-1})^2$

$\Longrightarrow$ quantum Casimir C generates the entire center Z_q and is also algebraically independent $\Rightarrow Z_q \simeq \mathbb{k}[C]$ ■

Remark: The structure of Z_q at roots of 1 is very different. Assume q is a primitive d^{th} root of 1, and define $e := \begin{cases} d, & \text{if } d\text{-odd} \\ d/2, & \text{if } d\text{-even.} \end{cases}$

Lemma 5: The elements $E^e, F^e, K^{\pm e}$ are in Z_q.
(simple exercise)

In fact, we have the following result (Problem 5 on Homework 2):

Theorem 2: The center Z_q of $U_q(\mathfrak{sl}_2)$, with q as above, is generated by $E^e, F^e, K^{\pm e}$ as well as C.

Hint: Try to adapt the proof of Theorem 1. In particular, consider $U_q'(\mathfrak{sl}_2)_0 := \{\sum_{i=0}^{e-1} F^i \cdot P_i \cdot E^i \mid P_i \in \mathbb{k}[K,K^{-1}]\}$. Then show:

1) Z_q is generated by E^e, F^e, $Z_q \cap U_q'(\mathfrak{sl}_2)_0$.
2) the restriction of Harish-Chandra homomorphism π to $Z_q \cap U_q'(\mathfrak{sl}_2)_0$ is injective (this allows to adapt Lemma 2 to the present setup)