Lecture #13

So far we used only the algebra structure on $U_q(\mathfrak{sl}_2)$ to:
- describe a basis of $U_q(\mathfrak{sl}_2)$
- center of $U_q(\mathfrak{sl}_2)$ for $q \neq \sqrt[?]{1}$
- f.d. dimensional representations of $U_q(\mathfrak{sl}_2)$ for $q \neq \sqrt[?]{1}$

However, what is really important is the Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ that allows to consider $\otimes$ of $U_q(\mathfrak{sl}_2)$-modules.

Theorem 1: $U_q(\mathfrak{sl}_2)$ has a Hopf algebra structure with coproduct Δ, counit ε, and antipode S uniquely determined by the following f-las:

$$\Delta: \ E \mapsto E\otimes 1 + K\otimes E, \ F \mapsto F\otimes K^{-1} + 1\otimes F, \ K \mapsto K\otimes K$$

$$S: \ E \mapsto -K^{-1}E, \ F \mapsto -FK, \ K \mapsto K^{-1}$$

$$\varepsilon: \ E \mapsto 0, \ F \mapsto 0, \ K \mapsto 1$$

► • First, we need to check that each of these 3 assignments is compatible with the defining rel-s of $U_q(\mathfrak{sl}_2)$:

$$K\cdot K^{-1} = 1 = K^{-1}K, \quad KE = q^2 EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K-K^{-1}}{q-q^{-1}}.$$

The first three are easy, so we will check only the last one

1) $\Delta(E)\Delta(F) - \Delta(F)\Delta(E) = [E\otimes 1 + K\otimes E, F\otimes K^{-1} + 1\otimes F] =$
$= EF\otimes K^{-1} + \cancel{E\otimes F} + \cancel{KF\otimes EK^{-1}} + K\otimes EF - FE\otimes K^{-1} - \cancel{FK\otimes K^{-1}E} - \cancel{E\otimes F} - K\otimes FE$
$= [E,F]\otimes K^{-1} + K\otimes [E,F] = \frac{K-K^{-1}}{q-q^{-1}}\otimes K^{-1} + K\otimes \frac{K-K^{-1}}{q-q^{-1}} = \frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}} = \frac{\Delta(K)-\Delta(K^{-1})}{q-q^{-1}}$ ✓

2) $[S(E), S(F)] = [-K^{-1}E, -FK] = K^{-1}EFK - FE = [E,F] = \frac{K-K^{-1}}{q-q^{-1}} = -\frac{S(K)-S(K^{-1})}{q-q^{-1}}$ ✓

3) $[\varepsilon(E), \varepsilon(F)] = 0 = \frac{\varepsilon(K)-\varepsilon(K^{-1})}{q-q^{-1}}$ ✓

• Next, one needs to check coassociativity and counity properties. It suffices to check these on generators, where it's straightforward.
[Exercise: Check all!

• Finally, we need to check that S is indeed an antipode, i.e.

$$\sum_{(x)} S(x')x'' = \sum_{(x)} x'S(x'') = \eta\varepsilon(x) \qquad \forall x \in U_q(\mathfrak{sl}_2)$$

But according to Lemma 1 of Lecture 3, it suffices to check this only for $x = E, F, K^{\pm 1}$. It's a straightforward check!
[Exercise: Check it!

Lecture #13

We next state two lemmas of computational nature.

Lemma 1: a) $S^2(u) = K^{-1}uK \quad \forall u \in U_q(sl_2)$

b) $S(E^r) = (-1)^r q^{r(r-1)} K^{-r} E^r$

$S(F^r) = (-1)^r q^{-r(r-1)} F^r K^r$

► a) As both S^2 and $u \mapsto K^{-1}uK$ are algebra homomorphisms $U_q(sl_2) \to U_q(sl_2)$, it suffices to check the above equality on the generators

$$S^2(K^{\pm 1}) = S(K^{\mp 1}) = K^{\pm 1} = K^{-1} \cdot K^{\pm 1} \cdot K$$

$$S^2(E) = S(-K^{-1}E) = -S(E)S(K^{-1}) = K^{-1} \cdot E \cdot K$$

$$S^2(F) = S(-FK) = -S(K)S(F) = K^{-1} \cdot F \cdot K$$

b) Prove both f-las by induction on r, or alternatively in a direct way:

$$S(E^r) = \underbrace{S(E) \cdot \_\_ \cdot S(E)}_{r} = \underbrace{(-K^{-1}E) \cdot \_\_ \cdot (-K^{-1}E)}_{r} = (-1)^r \cdot K^{-1}EK^{-1}E \ldots K^{-1}E$$

$$= (-1)^r \cdot q^{2 \cdot \frac{r(r-1)}{2}} \cdot K^{-r} E^r = (-1)^r q^{r(r-1)} K^{-r} E^r$$

$$S(F^r) = S(F) \cdot \_\_ \cdot S(F) = \underbrace{(-FK) \cdot \_\_ \cdot (-FK)}_{r} = (-1)^r q^{-2 \cdot \frac{r(r-1)}{2}} F^r K^{-r} = (-1)^r q^{-r(r-1)} F^r K^r$$

■

Lemma 2: The coproduct Δ behaves on the powers of generators as follows:

$$\Delta(K^n) = K^n \otimes K^n, \quad \Delta(E^n) = \sum_{i=0}^n q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} E^{n-i}K^i \otimes E^i, \quad \Delta(F^n) = \sum_{i=0}^n q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} F^i \otimes F^{n-i}K^{-i}$$

►
- First f-la is clear
- For the second one, note that $(K \otimes E) \cdot (E \otimes 1) = q^2 \cdot (E \otimes 1) \cdot (K \otimes E)$, hence we can apply q-binomial f-la (Proposition 3 from Lecture 6). To this end, we recall that $\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]_q!}{[i]_q! \, [n-i]_q!}$, $\binom{n}{i}_{q^2} = \frac{(n)_{q^2}!}{(i)_{q^2}! \, (n-i)_{q^2}!}$

As $[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{1-a} \cdot \frac{q^{2a}-1}{q^2-1} = q^{1-a} \cdot (a)_{q^2}$, we get $[a]_q! = (a)_{q^2}! \cdot q^{-\frac{a(a-1)}{2}}$ $\Rightarrow$

$$\Rightarrow \boxed{\binom{n}{i}_{q^2} = \begin{bmatrix} n \\ i \end{bmatrix} \cdot q^{\frac{n(n-1)}{2} - \frac{i(i-1)}{2} - \frac{(n-i)(n-i-1)}{2}} = q^{i(n-i)} \cdot \begin{bmatrix} n \\ i \end{bmatrix}}$$

So: $\Delta(E^n) = (E \otimes 1 + K \otimes E)^n = \sum_{i=0}^n \binom{n}{n-i}_{q^2} E^{n-i}K^i \otimes E^i = \sum_{i=0}^n q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} E^{n-i}K^i \otimes E^i$.

- Noting that $(F \otimes K^{-1}) \cdot (1 \otimes F) = q^2 \cdot (1 \otimes F) \cdot (F \otimes K^{-1})$, we likewise get:

$$\Delta(F^n) = (1 \otimes F + F \otimes K^{-1})^n = \sum_{i=0}^n \binom{n}{n-i}_{q^2} F^i \otimes F^{n-i}K^{-i} = \sum_{i=0}^n q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} F^i \otimes F^{n-i}K^{-i}$$

■

Lecture # 13

Exercise: Recalling the Hopf algebra structure on the universal enveloping algebras $U(\mathfrak{g})$ (from Theorem 1 of Lecture 7) and the exact degeneration $U_q(\mathfrak{sl}_2) \underset{q\to 1}{\rightsquigarrow} U(\mathfrak{sl}_2)$ (from Lemmas 2-3 of Lecture 8), verify that the degeneration of Δ, S, ε on $U_q(\mathfrak{sl}_2)$ recovers those on $U(\mathfrak{sl}_2)$

The goal for the rest of today is to construct the intertwiner of $\otimes$-product of fin. dimen. $U_q(\mathfrak{sl}_2)$-modules. To this end, we note:

1) given f.dim. vector spaces V_1 & V_2, the flip map $\tau: V_1\otimes V_2 \to V_2\otimes V_1$, $v_1\otimes v_2 \mapsto v_2\otimes v_1$ is a vector space isomorphism

2) given f.dim. $\mathfrak{sl}_2$-modules M_1 & M_2, the flip map $\tau: M_1\otimes M_2 \to M_2\otimes M_1$, $m_1\otimes m_2 \mapsto m_2\otimes m_1$ is actually an $\mathfrak{sl}_2$-module isomorphism since $\Delta = \underbrace{\Delta^{op}}_{=:\tau\circ\Delta}$

3) however given f.dim. $U_q(\mathfrak{sl}_2)$-modules M_1 & M_2, the flip map τ as above is NOT a $U_q(\mathfrak{sl}_2)$-module isomorphism. Our goal is to correct τ by extra terms to make it into a $U_q(\mathfrak{sl}_2)$-mod isom.

Assume: $q \overset{?}{\neq} \sqrt{1}$, char $\mathbb{k} \neq 2$.

Def: We define a sequence $\{\Theta_n\}_{n\geq -1}$ in $U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)$ by

$$\Theta_{-1}=0,\quad \Theta_0 := 1\otimes 1,\quad \Theta_n := a_n\cdot F^n\otimes E^n \text{ with } a_0=1,\ a_n = -q^{-(n-1)}\cdot\frac{q-q^{-1}}{[n]}a_{n-1}$$

so: $\boxed{a_n = (-1)^n\cdot \frac{(q-q^{-1})^n}{[n]_q!}\cdot q^{-\frac{n(n-1)}{2}}}$

Lemma 3: For any $n\in\mathbb{Z}_{\geq 0}$, we have

$$(K\otimes K)\Theta_n = \Theta_n(K\otimes K)$$
$$(E\otimes 1)\Theta_n + (K\otimes E)\Theta_{n-1} = \Theta_n(E\otimes 1) + \Theta_{n-1}(K^{-1}\otimes E)$$
$$(1\otimes F)\Theta_n + (F\otimes K^{-1})\Theta_{n-1} = \Theta_n(1\otimes F) + \Theta_{n-1}(F\otimes K)$$

▶ The first equality is obvious. For the second one, we note:

$[E\otimes 1, \Theta_n] = a_n\cdot [E,F^n]\otimes E^n = a_n\cdot [n]\cdot F^{n-1}[K;1-n]\otimes E^n$ (by Lemma 1 of Lecture 9)

$\Theta_{n-1}(K^{-1}\otimes E) - (K\otimes E)\Theta_{n-1} = a_{n-1}(F^{n-1}K^{-1}\otimes E^n - KF^{n-1}\otimes E^n) = a_{n-1}\cdot F^{n-1}(K^{-1}-q^{-2(n-1)}K)\otimes E^n$

Hence, the two expressions coincide b/c of the recursive f-la for a_n.

[**Exercise**: Check the 3^{d} equality in the same way

Lecture #13

Evoking Lemma 2 of Lecture 10, we note that under our assumptions on $q, \mathbb{K}$, any fin. dimen. $U_q(\mathfrak{sl}_2)$-module has a weight decomposition w.r.t. K-action, with weights being elts of the following set:

$$\boxed{\tilde{\Lambda} := \{\pm q^s \mid s \in \mathbb{Z}\}}$$

Let $f: \tilde{\Lambda} \times \tilde{\Lambda} \longrightarrow \mathbb{K}^*$ be a map s.t.

$$\boxed{f(\lambda, \mu) = \lambda f(\lambda, q^2 \mu) = \mu f(q^2 \lambda, \mu) \quad \forall \lambda, \mu \in \tilde{\Lambda}}$$

Exercise: Classify all such f

Now we are ready to construct an intertwiner b/w $\otimes$-products.
Explicitly, given two finite dimensional $U_q(\mathfrak{sl}_2)$-modules M_1, M_2, define

$$\tilde{f}: M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2$$

$$m_1 \otimes m_2 \longmapsto f(\lambda, \mu) \cdot m_1 \otimes m_2 \quad \forall \text{ weight } \lambda \text{ vector } m_1, \text{ weight } \mu \text{ vector } m_2$$

We also consider

$$\boxed{\Theta := \sum_{n \geq 0} \Theta_n : M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2}$$

which is well-defined since F, E act nilpotently on M_1, M_2 (Lemma 1 of Lect 10)

Finally, we consider the composition

$$\boxed{\Theta^f := \Theta \circ \tilde{f} : M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2}$$

Theorem 2: The map $\Theta^f \circ \tau : M_2 \otimes M_1 \longrightarrow M_1 \otimes M_2$ is a $U_q(\mathfrak{sl}_2)$-mod isom.

flip map