Lecture #14

We start by proving Theorem 2 from the end of Lecture #13.

Recall:

- $\Theta = \sum_{n\geq 0} \Theta_n$ with $\Theta_n = a_n \cdot F^n \otimes E^n$
- $f: \tilde{\Lambda}\times\tilde{\Lambda} \to \mathbb{k}^*$ satisfies $f(\lambda,\mu) = \lambda f(\lambda, q^2\mu) = \mu f(q^2\lambda,\mu)$
  $\rightsquigarrow \tilde{f}: M_1\otimes M_2 \to M_1\otimes M_2$ via $m_1\otimes m_2 \mapsto f(\lambda,\mu)\cdot m_1\otimes m_2 \quad \forall m_1\in(M_1)_\lambda,\ m_2\in(M_2)_\mu$.
- Combining the above two we get $\Theta^f := \Theta\circ\tilde{f}: M_1\otimes M_2 \to M_1\otimes M_2$

Theorem 1: The map $\Theta^f\circ\underset{\text{swap}}{\tau}: M_2\otimes M_1 \to M_1\otimes M_2$ is an isomorphism of $U_q(\mathfrak{sl}_2)$-modules

▸

- Clearly $\tilde{f}$ and τ are bijective, whereas $F\otimes E$ acts nilpotently and so Θ is bijective too. Therefore, $\Theta^f\circ\tau$ is a vector space isomorphism.
- It remains to check that $\Theta^f\circ\tau$ intertwines $U_q(\mathfrak{sl}_2)$-actions, i.e.
$$\boxed{\Delta(x)\circ\Theta^f = \Theta^f\circ\Delta^{op}(x) \quad \forall x\in U_q(\mathfrak{sl}_2)} \quad (*)$$
  where $\Delta^{op}(x) := \tau(\Delta(x))$ appears naturally due to
$$\tau(x(m)) = \tau(\Delta(x)m) = \Delta^{op}(x)(\tau(m)) \quad \forall m\in M_2\otimes M_1,\ x\in U_q(\mathfrak{sl}_2)$$
  Moreover, it suffices to verify $(*)$ for the generators $E, F, K^{\pm1}$.
- Evoking $\Theta = \sum_{n\geq 0}\Theta_n$ and Lemma 3 from Lecture #13, we find:
$$\Delta(K)\Theta = \Theta\cdot\tilde{\Delta}(K) \quad \text{with} \quad \tilde{\Delta}(K) = K\otimes K$$
$$\Delta(E)\Theta = \Theta\cdot\tilde{\Delta}(E) \quad \text{with} \quad \tilde{\Delta}(E) = E\otimes 1 + K^{-1}\otimes E$$
$$\Delta(F)\Theta = \Theta\cdot\tilde{\Delta}(F) \quad \text{with} \quad \tilde{\Delta}(F) = 1\otimes F + F\otimes K.$$
  Therefore, to verify $(*)$ for $x = E, F, K^{\pm1}$, it suffices to verify
$$\boxed{\tilde{\Delta}(x)\circ\tilde{f} = \tilde{f}\circ\Delta^{op}(x) \quad \text{for } x = E, F, K^{\pm1}} \quad (**)$$
  To this end, pick any $m_1\in(M_1)_\lambda$, $m_2\in(M_2)_\mu$.
  1) As $Km_1 = \lambda\cdot m_1$, $Km_2 = \mu\cdot m_2$, the $x = K^{\pm1}$ case of $(**)$ follows.
  2) $(E\otimes 1 + K^{-1}\otimes E)\tilde{f}(m_1\otimes m_2) = f(\lambda,\mu)\, Em_1\otimes m_2 + \lambda^{-1} f(\lambda,\mu)\, m_1\otimes Em_2$
     $\tilde{f}(E\otimes K + 1\otimes E)(m_1\otimes m_2) = \tilde{f}(Em_1\otimes m_2\cdot\mu + m_1\otimes Em_2) = \mu f(\lambda q^2,\mu)\cdot Em_1\otimes m_2 + f(\lambda,\mu q^2)\cdot m_1\otimes Em_2$
     Hence, $x = E$ case of $(**)$ follows from 2 eq-s on f
  3) $(1\otimes F + F\otimes K)\tilde{f}(m_1\otimes m_2) = f(\lambda,\mu)\cdot m_1\otimes Fm_2 + \mu f(\lambda,\mu)\cdot Fm_1\otimes m_2$
     $\tilde{f}(F\otimes 1 + K^{-1}\otimes F)(m_1\otimes m_2) = f(q^{-2}\lambda,\mu)\cdot Fm_1\otimes m_2 + \lambda^{-1} f(\lambda, q^{-2}\mu)\, m_1\otimes Fm_2$.
     ⟫ due to two difference eq-s on $f: \tilde{\Lambda}\times\tilde{\Lambda}\to\mathbb{k}$

Remark: In the above proof we witnessed three related homomorphisms

1) Δ: $K \mapsto K\otimes K$, $E \mapsto E\otimes 1 + K\otimes E$, $F \mapsto F\otimes K^{-1} + 1\otimes F$

2) Δ^{op}: $K \mapsto K\otimes K$, $E \mapsto E\otimes K + 1\otimes E$, $F \mapsto F\otimes 1 + K^{-1}\otimes F$

3) $\tilde{\Delta}$: $K \mapsto K\otimes K$, $E \mapsto E\otimes 1 + K^{-1}\otimes E$, $F \mapsto F\otimes K + 1\otimes F$

Moreover, as verified in the proof above, they satisfy:

$$\left.\begin{array}{l}\Delta(x)\, H = H\, \tilde{\Delta}(x) \\ \tilde{\Delta}(x)\, \tilde{f} = \tilde{f}\, \Delta^{op}(x)\end{array}\right\} \Rightarrow \Delta(x)\, H^f = H^f\, \Delta^{op}(x)$$

which is exactly the statement of Theorem 1

Note: $\tilde{\Delta}(x) = (\sigma\otimes\sigma)\,\Delta(\sigma^{-1}(x))$, $\sigma: U_q(sl_2) \to U_q(sl_2)^{op}$ given by $E\mapsto E$, $F\mapsto F$, $K^{\pm 1}\mapsto K^{\mp 1}$

Remark: We note that an abstract isomorphism $M_1\otimes M_2 \simeq M_2\otimes M_1$ as $U_q(sl_2)$-mod for finite dimensional M_1, M_2 follows from the weight decomposition of both (together with complete reducibility of fin. dim. $U_q(sl_2)$-mod and explicit description of irreducible modules $L(n,\pm)$ and their weight spaces).

However, the explicit construction of Theorem 1 provides a functorial isomorphism, i.e. given any $U_q(sl_2)$-$mod_{f.d.}$ morphisms $M_1 \xrightarrow{\varphi_1} N_1$ and $M_2 \xrightarrow{\varphi_2} N_2$ we get a commutative diagram

$$\begin{array}{ccc} M_2\otimes M_1 & \xrightarrow{H^f\circ\tau} & M_1\otimes M_2 \\ \downarrow{\scriptstyle \varphi_2\otimes\varphi_1} & & \downarrow{\scriptstyle \varphi_1\otimes\varphi_2} \\ N_2\otimes N_1 & \xrightarrow{H^f\circ\tau} & N_1\otimes N_2 \end{array}$$

Remark: We note that if the above was true for any M_1, M_2 (not just fin. dim.) then the above functoriality would yield that all these isomorphism arise naturally from a single one for $M_1\simeq M_2\simeq U_q(sl_2)$. In the latter case, the image of $1\otimes 1 \in M_2\otimes M_1$, denoted by R, would satisfy

$$\Delta(x)\, R = R\, \Delta^{op}(x) \quad \forall x\in U_q(sl_2)$$

In our setup, there is no such R which explains our exposition.
However, in the formal setup (i.e. $q = e^{\frac{\hbar}{2}}$ and we work over $k[[\hbar]]$ not k), such R does exist and goes back to Drinfeld

We conclude with another property of Θ^f. Pick any 3 fin. dim. $U_q(sl_2)$-modules M_1, M_2, M_3, and consider

$$\Theta^f_{12} := \Theta^f \otimes \mathrm{id}, \quad \Theta^f_{23} := \mathrm{id} \otimes \Theta^f, \quad \Theta^f_{13} := (\mathrm{id} \otimes \tau)(\Theta^f \otimes \mathrm{id})(\mathrm{id} \otimes \tau) : M_1 \otimes M_2 \otimes M_3 \circlearrowleft$$

Theorem 2: We have the following equality:

$$\Theta^f_{12} \circ \Theta^f_{13} \circ \Theta^f_{23} = \Theta^f_{23} \circ \Theta^f_{13} \circ \Theta^f_{12}$$

[Remark: When $M_1 = M_2 = M_3$, we get a solution of so-called quantum Yang-Baxter eq-n.

▸ Evoking that $\Theta^f = \Theta \circ f$, we see that each side of the above equality is:

$$\text{LHS} = \Theta_{12} \circ \tilde{f}_{12} \circ \Theta_{13} \circ \tilde{f}_{13} \circ \Theta_{23} \circ \tilde{f}_{23}$$

$$\text{RHS} = \Theta_{23} \circ \tilde{f}_{23} \circ \Theta_{13} \circ \tilde{f}_{13} \circ \Theta_{12} \circ \tilde{f}_{12}$$

We shall first move all $\tilde{f}_{??}$ to the rightmost part using the following lemmas. To state the first of them, we define

$$\Theta' := \sum_{n \geq 0} \Theta'_n \quad \text{with } \Theta'_n = a_n \cdot F^n \otimes K^n \otimes E^n$$

$$\Theta'' := \sum_{n \geq 0} \Theta''_n \quad \text{with } \Theta''_n = a_n \cdot F^n \otimes K^{-n} \otimes E^n$$

Lemma 1: a) $\tilde{f}_{12} \circ \Theta_{13} = \Theta' \circ \tilde{f}_{12}$

b) $\tilde{f}_{23} \circ \Theta_{13} = \Theta'' \circ \tilde{f}_{23}$

▹ a) Pick $m = m_1 \otimes m_2 \otimes m_3 \in (M_1)_\lambda \otimes (M_2)_\mu \otimes (M_3)_\nu$. Then

$$\tilde{f}_{12} \circ \Theta_{13}(m) = \sum_{n \geq 0} a_n f(\lambda q^{-2n}, \mu) \cdot F^n m_1 \otimes m_2 \otimes E^n m_3 \underset{\text{eq-n on } f}{=\!=} \sum_{n \geq 0} a_n \cdot f(\lambda, \mu) \cdot \mu^n \cdot F^n m_1 \otimes m_2 \otimes E^n m_3$$

$$= \sum_{n \geq 0} f(\lambda, \mu) \cdot a_n \cdot F^n m_1 \otimes K^n m_2 \otimes E^n m_3 = \Theta' \circ \tilde{f}_{12}(m)$$

b) Pick m as above. Then:

$$\tilde{f}_{23} \circ \Theta_{13}(m) = \sum_{n \geq 0} a_n f(\mu, \nu q^{2n}) F^n m_1 \otimes m_2 \otimes E^n m_3 \underset{\text{eq-n on } f}{=\!=} \sum_{n \geq 0} a_n \cdot f(\mu, \nu) \cdot \mu^{-n} \cdot F^n m_1 \otimes m_2 \otimes E^n m_3$$

$$= f(\mu, \nu) \cdot \sum_{n \geq 0} a_n \cdot F^n m_1 \otimes K^{-n} m_2 \otimes E^n m_3 = \Theta'' \circ \tilde{f}_{23}(m)$$

□

Lemma 2: a) $\tilde{f}_{12} \tilde{f}_{13} \Theta_{23} = \Theta_{23} \tilde{f}_{12} \tilde{f}_{13}$

b) $\tilde{f}_{23} \tilde{f}_{13} \Theta_{12} = \Theta_{12} \tilde{f}_{23} \tilde{f}_{13}$

▸ Same argument as in the proof of Lemma 1. For example, for part a), use:

$$\tilde{f}_{12} \tilde{f}_{13} \Theta_{23}(m) = \sum_{n \geq 0} a_n f(\lambda, \mu q^{-2n}) f(\lambda, \nu q^{2n}) \cdot m_1 \otimes F^n m_2 \otimes E^n m_3 \overset{2 \text{ eq-s on } f}{=\!=} \sum_{n \geq 0} a_n f(\lambda, \mu) f(\lambda, \nu) m_1 \otimes F^n m_2 \otimes E^n m_3 = \Theta_{23} \tilde{f}_{12} \tilde{f}_{13}(m)$$

□

Lecture #14

Combining Lemmas 1-2 with the obvious commutativity $\tilde{f}_{12}\tilde{f}_{13}\tilde{f}_{23} = \tilde{f}_{23}\tilde{f}_{13}\tilde{f}_{12}$, it thus remains to prove:

$$\boxed{\Theta_{12} \circ \Theta' \circ \Theta_{23} = \Theta_{23} \circ \Theta'' \circ \Theta_{12}}$$

Lemma 3: $\Theta_{12}\Theta'\Theta_{23} = \Theta_{23}\Theta''\Theta_{12}$

▷ $\Theta_{23} \circ \Theta'' = \sum\limits_{i,j\geq 0} a_i a_j (1\otimes F^j \otimes E^j)(F^i \otimes K^{-i} \otimes E^i) = \sum\limits_{n\geq 0} \left(\sum\limits_{i=0}^{n} a_i a_{n-i} \cdot F^i \otimes F^{n-i}K^{-i}\right) \otimes E^n$

[Exercise: Check $a_i a_{n-i} = a_n \cdot q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}$

Recall: $\sum\limits_{i=0}^{n} q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} F^i \otimes F^{n-i}K^{-i} = \Delta(F^n)$

$\Rightarrow$ $\boxed{\Theta_{23} \circ \Theta'' = (\Delta \otimes id)(\Theta)}$

Since $\Delta(x)\Theta = \Theta\tilde{\Delta}(x)$, with $\tilde{\Delta}$ being defined in the proof of Theorem 1, we get:

$$\boxed{RHS = \Theta_{23} \circ \Theta'' \circ \Theta_{12} = \Theta_{12} \circ (\tilde{\Delta} \otimes id)(\Theta)}$$

Thus, to prove the above lemma, it remains to show $(\tilde{\Delta} \otimes id)(\Theta) = \Theta' \circ \Theta_{23}$.

But, the composition $\Theta' \circ \Theta_{23}$ can be evaluated as above:

$$\Theta' \circ \Theta_{23} = \sum_{i,j\geq 0} a_i a_j \, F^i \otimes K^i F^j \otimes E^{i+j} = \sum_{n\geq 0} \Big(\underbrace{\sum_{i=0}^{n} a_i a_{n-i} \cdot F^i \otimes K^i F^{n-i}}_{\overset{\text{Exercise}}{=} a_n \cdot \tilde{\Delta}(F^n)} \Big) \otimes E^n =$$

$$= (\tilde{\Delta} \otimes id)(\Theta)$$

□

This completes our proof of the Theorem.

■