Lecture #15

- Last time:

1) $\Theta^f \circ \tau : M_2 \otimes M_1 \xrightarrow{\sim} M_1 \otimes M_2$ as $U_q(sl_2)$-mod

2) $\Theta^f_{12} \circ \Theta^f_{13} \circ \Theta^f_{23} = \Theta^f_{23} \circ \Theta^f_{13} \circ \Theta^f_{12} : M_1 \otimes M_2 \otimes M_3 \circlearrowleft$

Given that f satisfies two difference equations:
$\boxed{f(\lambda,\mu) = \lambda f(\lambda, q^2\mu) = \mu f(q^2\lambda, \mu)}$

Remark: a) There are many maps $f : \Lambda \times \Lambda \to \mathbb{k}^*$ satisfying these 2 difference eq-s (exercise: classify all).

b) In the simplest case of $M_1 = M_2 = L(1,+)$, it's easy to see that the map $\Theta : L(1,+)^{\otimes 2} \circlearrowleft$ in the basis $\{v_0\otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ is given by

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1}-q & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) Since the weights of $L(1,+)$ are $q^{\pm 1}$ (as $Kv_0 = q\cdot v_0$, $Kv_1 = q^{-1}\cdot v_1$), we see that f is uniquely determined by $\lambda := f(q,q)$, since $f(q,q^{-1}) = q\cdot f(q,q) = f(q^{-1},q)$ and $f(q^{-1},q^{-1}) = q^{-1}\cdot f(q^{-1},q)$ so that $f(q,q^{-1}) = q\cdot\lambda$, $f(q^{-1},q) = q\cdot\lambda$, $f(q^{-1},q^{-1}) = \lambda$. Thus:

$$\Theta^f = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda q & 0 & 0 \\ 0 & \lambda q(q^{-1}-q) & \lambda q & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

The following result allows to relate $U_q(sl_2)$-mod$_{f.d.}$ to tensor categories:

Proposition 1: Let M_1, M_2, M_3 be fin. dim. $U_q(sl_2)$-modules. We also assume that f in addition is multiplicative in each argument:

$$f(\lambda, \mu\nu) = f(\lambda,\mu) f(\lambda,\nu), \quad f(\lambda\mu, \nu) = f(\lambda,\nu) f(\mu,\nu)$$

for all weights λ, μ, ν of these modules. Then:

$$\begin{tikzcd}
 & M_1 \otimes (M_3 \otimes M_2) \arrow[r, "can"] & (M_1 \otimes M_3) \otimes M_2 \arrow[dr, "\check{R}\otimes id"] & \\
M_1 \otimes (M_2 \otimes M_3) \arrow[ur, "id\otimes\check{R}"] \arrow[dr, "can"'] & & & (M_3 \otimes M_1) \otimes M_2 \\
 & (M_1 \otimes M_2) \otimes M_3 \arrow[r, "\check{R}"] & M_3 \otimes (M_1 \otimes M_2) \arrow[ur, "can"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (M_2 \otimes M_1) \otimes M_3 \arrow[r, "can"] & M_2 \otimes (M_1 \otimes M_3) \arrow[dr, "id\otimes\check{R}"] & \\
(M_1 \otimes M_2) \otimes M_3 \arrow[ur, "\check{R}\otimes id"] \arrow[dr, "can"'] & & & M_2 \otimes (M_3 \otimes M_1) \\
 & M_1 \otimes (M_2 \otimes M_3) \arrow[r, "\check{R}"] & (M_2 \otimes M_3) \otimes M_1 \arrow[ur, "can"'] &
\end{tikzcd}$$

where $can : (A\otimes B)\otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ is the canonical associativity map $(a\otimes b)\otimes c \mapsto a \otimes (b\otimes c)$,
$\check{R} := \Theta^f \circ \tau : B \otimes A \xrightarrow{\sim} A \otimes B$ is the isom. of $U_q(sl_2)$-mod$_{f.d.}$

Proof of Proposition 1

The "upper" composition in the first diagram reads:

$$(H)_{12}\circ\tilde{f}_{12}\circ\tau_{12}\circ(H)_{23}\circ\tilde{f}_{23}\circ\tau_{23}=(H)_{12}\circ\tilde{f}_{12}\circ(H)_{13}\circ\tau_{12}\circ\tilde{f}_{23}\circ\tau_{23}=(H)_{12}\circ\tilde{f}_{12}\circ(H)_{13}\circ\tilde{f}_{13}\circ\tau_{12}\circ\tau_{23}$$

But, due to Lemma 1 of Lecture 14, we have:

$$\tilde{f}_{12}\circ(H)_{13}=(H)'\circ\tilde{f}_{12}$$

Finally, by analogy with the proof of Lemma 3 of Lecture 14, we have:

$$(H)_{12}\circ(H)'=(1\otimes\Delta)(H).$$

[Exercise: Prove this.

Thus, the upper composition of the 1st diagram reads:

$$\boxed{(1\otimes\Delta)(H)\circ(\tilde{f}_{12}\circ\tilde{f}_{13})\circ(\tau_{12}\circ\tau_{23})}$$

On the other hand, evaluating the lower composition of the 1st diagram on the tensor product of weight vectors $m_1\otimes m_2\otimes m_3\in(M_1)_\lambda\otimes(M_2)_\mu\otimes(M_3)_\nu$ we get:

$$\boxed{m_1\otimes(m_2\otimes m_3)\xrightarrow{\text{can}}\underbrace{(m_1\otimes m_2)}_{\in\,(M_1\otimes M_2)_{\lambda\mu}}\otimes m_3\xmapsto{\tau}m_3\otimes(m_1\otimes m_2)\xmapsto{(H)^f}f(\nu,\lambda\mu)\circ\big((1\otimes\Delta)(H)\big)(m_3\otimes(m_1\otimes m_2))}$$

As $\tau_{12}\circ\tau_{23}: m_1\otimes m_2\otimes m_3\longmapsto m_3\otimes m_1\otimes m_2$

$\tilde{f}_{12}\circ\tilde{f}_{13}: m_3\otimes m_1\otimes m_2\longmapsto f(\nu,\lambda)\cdot f(\nu,\mu)\cdot m_3\otimes m_1\otimes m_2$

we conclude that the 1st diagram is indeed commutative due to

$$f(\nu,\lambda\mu)=f(\nu,\lambda)f(\nu,\mu).$$

Likewise, the 2nd diagram is commutative due to $f(\lambda\mu,\nu)=f(\lambda,\nu)f(\mu,\nu)$ ∎

Def: A finite dimensional $U_q(sl_2)$-module is called of type 1 if all eigenvalues of K are powers of q (no minus sign!)

Let $\Lambda=\{q^s\mid s\in\mathbb{Z}\}\subseteq\bar{\Lambda}=\{\pm q^s\mid s\in\mathbb{Z}\}$.

Exercise: a) Classify all $f:\Lambda\times\Lambda\to\mathbb{K}^*$ satisfying 4 conditions

$$f(\lambda,\mu)=\lambda f(\lambda,q^2\mu)=\mu f(q^2\lambda,\mu),\quad f(\lambda\mu,\nu)=f(\lambda,\nu)f(\mu,\nu),\quad f(\lambda,\mu\nu)=f(\lambda,\mu)f(\lambda,\nu)$$

b) Classify all $f:\bar{\Lambda}\times\bar{\Lambda}\to\mathbb{K}^*$ satisfying the above 4 condition

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Hint to Exercise

a) Combine $f(q^r, q^s) = f(q,q)^{rs}$ with $f(q,q)^2 = q^{-1} f(q,1)$ to deduce $f(q^r, q^s) = (q^{-1/2})^{rs}$ given $\mathbb{K}$ contains a square root of q.

b) Combine $f(-1,1)^2 = f(-1,1)$ with $f(1,q) = f(-1,q)^2 = -f(-1,1)$ to get a contradiction.

Remark: The absence of $f: \Lambda \times \Lambda \to \mathbb{K}^*$ satisfying these 4 conditions explains why people usually consider only type 1 fin.dim. $U_q(\mathfrak{sl}_2)$-modules in the relation to braided tensor categories. The latter are the categories $\mathcal{C}$ equipped with functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$
$$(X, Y) \mapsto X \otimes Y$$
together with functorial isomorphisms (called associativity/commutativity constants)
$$(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \quad \text{and} \quad X \otimes Y \xrightarrow{\sim} Y \otimes X.$$
subject to some compatibilities, including the two diagrams from Prop 1, called the hexagonal diagrams.

For the rest of today (and next lecture) we will try to answer:

Question: What is the exact relation between the quantum groups $M_q(2)$, $SL_q(2)$ that we discussed in the first 2 weeks and $U_q(\mathfrak{sl}_2)$ that we have been discussing over the last 2 weeks?

To get an answer, we shall first understand the classical $(q=1)$ case. To this end, we start with the following definition.

Def: a) Given two bialgebras A and B, a bilinear form $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{K}$ is called a bialgebra pairing if the following holds (using Sweedler's notation):

$$\boxed{\begin{array}{l} \langle a, b_1 b_2 \rangle = \sum_{(a)} \langle a', b_1 \rangle \langle a'', b_2 \rangle , \quad \langle a_1 a_2, b \rangle = \sum_{(b)} \langle a_1, b'' \rangle \langle a_2, b' \rangle \\ \langle a, 1_B \rangle = \varepsilon_A(a), \quad \langle 1_A, b \rangle = \varepsilon(b) \end{array}}$$

b) If in addition A, B are Hopf algebras, then such $\langle \cdot, \cdot \rangle$ is a Hopf pairing if also
$$\boxed{\langle S_A(a), b \rangle = \langle a, S_B^{-1}(b) \rangle}$$

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Remark: The above definition differs slightly from [Kassel, Def V.7.1]. To get his setup, take $A = U$, $B = H^{cop}$.

The following simple result explains the origin of this definition

Lemma 1: a) Given bialgebras A, B and a bilinear pairing $\langle , \rangle : A \times B \to \mathbb{k}$, $\langle , \rangle$ is a bialgebra pairing iff the natural linear maps

$$\varphi : A^{op} \to B^* \qquad \varphi(a)(b) = \langle a, b \rangle$$
$$\psi : B \to A^* \qquad \psi(b)(a) = \langle a, b \rangle$$

are algebra morphisms.

b) If $\dim(B) < \infty$, then $\langle , \rangle$ - bialgebra pairing iff above $\varphi : A^{op} \to B^*$ is a bialgebra morphism

▶ a) Let's show that $\langle a_1 a_2, b \rangle = \sum_{(b)} \langle a_1, b'' \rangle \langle a_2, b' \rangle$ is equivalent to $\varphi(a_1 a_2) = \varphi(a_2)\varphi(a_1)$

Indeed:

$\varphi(a_1 a_2)(b) = \langle a_1 a_2, b \rangle$ while $(\varphi(a_2)\varphi(a_1))(b) = \sum_{(b)} \langle a_2, b' \rangle \cdot \langle a_1, b'' \rangle$.

Also: $\varphi(1_A)(b) = \langle 1_A, b \rangle = \varepsilon_B(b)$ implies $\varphi(1_A) = 1_{B^*}$ and vice-versa.

Likewise, $\langle a, b_1 b_2 \rangle = \sum_{(a)} \langle a', b_1 \rangle \langle a'', b_2 \rangle$ is equivalent to $\psi(b_1 b_2) = \psi(b_1)\psi(b_2)$, while $\langle a, 1_B \rangle = \varepsilon_A(a)$ is equivalent to $\psi(1_B) = 1_{A^*}$

b) It suffices to check that $\begin{cases} \langle a, b_1 b_2 \rangle = \sum_{(a)} \langle a', b_1 \rangle \langle a'', b_2 \rangle \\ \langle a, 1_B \rangle = \varepsilon_A(a) \end{cases}$ are equivalent to φ being a coalgebra morphism. The first follows from

$$\Delta(\varphi(a))(b \otimes c) = \varphi(a)(bc) = \langle a, bc \rangle = \sum_{(a)} \langle a', b \rangle \langle a'', c \rangle = (\varphi \otimes \varphi)(\Delta(a))(b \otimes c)$$

while the 2nd is also clear ■

Next time: construct a bialgebra pairing $M(2) \times U(sl_2) \to \mathbb{k}$ which gives rise to a Hopf pairing $SL(2) \times U(sl_2) \to \mathbb{k}$. Both will naturally generalize to q-version.

Remark: A geometric way to think about the above pairing b/w $\mathcal{O}(SL_2)$ and $U(\mathfrak{sl}_2)$ is by realizing $U(\mathfrak{sl}_2)$ = left/right-invariant differential operators on $SL_2 = \mathrm{Spec}(\mathcal{O}(SL_2))$ and evaluating the application of this operator to f-n of $\mathcal{O}(SL_2)$ at $1 \in SL_2$