Lecture #16

Last time we introduced the notion of bialgebra pairings

$$\langle\, ,\, \rangle : A \times B \longrightarrow \Bbbk$$

and proved that it is equivalent to natural linear maps $\varphi : A^{op} \to B^*$, $\psi : B \to A^*$ (induced by $\langle\, ,\, \rangle$) being algebra morphisms. We shall now apply this in the setup $A = M(2)$, $B = U(\mathfrak{sl}_2)$. Note that A - commutative (it's a polynomial algebra) while $U(\mathfrak{sl}_2)$ is cocommutative $\Rightarrow U(\mathfrak{sl}_2)^*$ is commutative.

Consider the 2-dim irreducible representation V_1 of $\mathfrak{sl}_2$, which gives rise to

$$\rho_{V_1} : U(\mathfrak{sl}_2) \longrightarrow \mathrm{End}(V_1) = \mathrm{Mat}_{2\times 2}(\Bbbk)$$
$$u \longmapsto \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Proposition 1: Consider the algebra morphism $\varphi : M(2)^{op} \to U(\mathfrak{sl}_2)^*$ defined via
$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$
and define the pairing $\langle\, ,\, \rangle : M(2) \times U(\mathfrak{sl}_2) \to \Bbbk$ via $\langle h, u\rangle = \varphi(h)(u)$.
Then: $\langle\, ,\, \rangle$ is a bialgebra pairing.

▶ As $M(2)^{op} \simeq M(2) \simeq \Bbbk[a,b,c,d]$ and $U(\mathfrak{sl}_2)^*$ - commutative, the above assignment does uniquely extends to an algebra morphism $M(2)^{op} \to U(\mathfrak{sl}_2)^*$. The latter is equivalent to the following 2 properties (as shown in Lemma 1 of Lecture 15):

$$\boxed{\langle h_1 h_2, u\rangle = \sum_{(u)} \langle h_1, u''\rangle \langle h_2, u'\rangle \quad , \quad \langle 1, u\rangle = \varepsilon(u) \quad \forall\ h_1, h_2 \in M(2),\ u \in U(\mathfrak{sl}_2)}$$

We shall now check the validity of the other two properties of bialgebra pairings:

- Note that $\rho_{V_1}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \langle a,1\rangle & \langle b,1\rangle \\ \langle c,1\rangle & \langle d,1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Combining this with ε and $\langle -, 1\rangle$ being algebra homomorphisms (the latter follows from the property we already checked as $\langle h_1 h_2, 1\rangle = \langle h_1, 1\rangle \langle h_2, 1\rangle$) we conclude that

$$\boxed{\langle h, 1\rangle = \varepsilon(h) \quad \forall\, h \in M(2)}$$

- We shall now verify the remaining property of bialgebra pairings:

$$(*) \quad \boxed{\langle h, u_1 u_2\rangle = \sum_{(h)} \langle h', u_1\rangle \cdot \langle h'', u_2\rangle \quad \forall\, h \in M(2),\ u_1, u_2 \in U(\mathfrak{sl}_2)}$$

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(Continuation of the proof)

Know: $\rho_{V_1}(u_1 u_2) = \rho_{V_1}(u_1)\cdot \rho_{V_1}(u_2)\quad \forall u_1,u_2\in U(sl_2)$

$$\Downarrow$$

$$\begin{pmatrix} \langle a,u_1u_2\rangle & \langle b,u_1u_2\rangle \\ \langle c,u_1u_2\rangle & \langle d,u_1u_2\rangle \end{pmatrix} = \begin{pmatrix} \langle a,u_1\rangle & \langle b,u_1\rangle \\ \langle c,u_1\rangle & \langle d,u_1\rangle \end{pmatrix}\cdot \begin{pmatrix} \langle a,u_2\rangle & \langle b,u_2\rangle \\ \langle c,u_2\rangle & \langle d,u_2\rangle \end{pmatrix}$$

$$\Downarrow \quad \Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

(*) holds for $h\in\{a,b,c,d\}$ and any $u_1,u_2\in U(sl_2)$

The desired equality (*) $\forall h,u_1,u_2$ now follows from the following simple:

Lemma 1: If (*) holds for $h\in\{h_1,h_2\}$ and arbitrary u_1,u_2, then it also holds for $h=h_1\cdot h_2$ and arbitrary u_1,u_2.

$\triangleright$ $\langle h_1h_2, u_1u_2\rangle = \sum_{(u_1u_2)} \langle h_1,(u_1u_2)''\rangle\cdot\langle h_2,(u_1u_2)'\rangle = \sum_{(u_1)(u_2)} \langle h_1, u_1''\cdot u_2''\rangle\cdot\langle h_2, u_1'\cdot u_2'\rangle$

|| assumption

$\sum_{(h_1h_2)} \langle (h_1h_2)',u_1\rangle\langle (h_1h_2)'',u_2\rangle = \sum_{(h_1)(h_2)} \langle h_1'h_2',u_1\rangle\cdot\langle h_1''h_2'',u_2\rangle = \sum_{(u_1)(u_2)(h_1)(h_2)} \langle h_1',u_1''\rangle\cdot\langle h_1'',u_2''\rangle\cdot\langle h_2',u_1'\rangle\cdot\langle h_2'',u_2'\rangle$ $\square$

This concludes our proof of the Proposition ▥

Lemma 2: $\varphi(\det - 1) = 0$

$\blacktriangleright$ Suffices to verify $\langle \det -1, u\rangle = 0\ \forall u\in U(sl_2)$, which is equivalent to:

$\langle ad-bc, u\rangle = \varepsilon(u)\ \forall u\in U(sl_2)$

Since $\det = ad-bc$ is group-like ($\Delta(\det)=\det\otimes\det$) and (*) holds, it suffices to verify the above for $u\in\{1,e,f,h\}$. The latter is straightforward:

- $\langle ad-bc, e\rangle = \langle a\otimes d - b\otimes c, 1\otimes e + e\otimes 1\rangle = \underbrace{\varepsilon(a)}_{=1}\cdot\langle d,e\rangle - \underbrace{\varepsilon(b)}_{=0}\cdot\langle c,e\rangle +$
  $+\underbrace{\varepsilon(d)}_{=1}\cdot\langle a,e\rangle - \underbrace{\varepsilon(c)}_{=0}\cdot\langle b,e\rangle = \langle d,e\rangle + \langle a,e\rangle = 0$ as $\rho_{V_1}(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$ $\langle a,e\rangle$ $\langle d,e\rangle$
- $\langle ad-bc, f\rangle \overset{\text{likewise}}{=\!=} \langle d,f\rangle + \langle a,f\rangle = 0$ as $\rho_{V_1}(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$
- $\langle ad-bc, h\rangle \overset{\text{likewise}}{=\!=} \langle d,h\rangle + \langle a,h\rangle = 0$ as $\rho_{V_1}(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$
- $\langle ad-bc, 1\rangle = \varepsilon(\det)$ ▥

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Remark: In the above proof we used conventions that given a bilinear pairing $(\,,\,): U\times V \to \mathbb{k}$ of vector spaces, one also gets a bilinear pairing $U^{\otimes 2}\times V^{\otimes 2}\to \mathbb{k}$ defined via

$$(u_1\otimes u_2,\ v_1\otimes v_2) := (u_1,v_1)\cdot(u_2,v_2).$$

In particular, the first two properties of bialgebra pairings $\langle\,,\,\rangle: A\times B\to \mathbb{k}$ can rather be written as follows:

$$\boxed{\begin{array}{ll} \langle a, b_1\cdot b_2\rangle = \langle \Delta(a), b_1\otimes b_2\rangle & \forall a,a_1,a_2\in A \\ \langle a_1\cdot a_2, b\rangle = \langle a_2\otimes a_1, \Delta(b)\rangle & b,b_1,b_2\in B \end{array}}$$

According to Lemma 2, the algebra morphism $\varphi: M(2)^{op}\to U(\mathfrak{sl}_2)^*$ factors through

$$\varphi:\ SL(2)^{op} = M(2)^{op}/(ad-bc-1) \longrightarrow U(\mathfrak{sl}_2)^*.$$

Proposition 2: The resulting pairing $\langle\,,\,\rangle: SL(2)\times U(\mathfrak{sl}_2)\to \mathbb{k}$, $(h, u)\mapsto \varphi(h)(u)$ is Hopf

▷ First, we note that $\langle\,,\,\rangle$ is indeed a bialgebra pairing. As φ is an algebra morphism, it suffices to check that $\psi: U(\mathfrak{sl}_2)\to SL(2)^*$ is also algebra morphism (see Lemma 1 of Lecture 15). However, composing it with embedding $SL(2)^*\hookrightarrow M(2)^*$, we obtain $\psi: U(\mathfrak{sl}_2)\to M(2)^*$ which is an algebra morphism by Proposition 1 (and again Lemma 1 of Lecture 15).

Hence, it remains to verify compatibility of $\langle\,,\,\rangle$ with antipodes:

$$(**)\qquad \boxed{\langle S(h), u\rangle = \langle h, S^{-1}(u)\rangle}$$

First, let us check $(**)$ on the generators ($h\in\{a,b,c,d\}$, $u\in\{e,f,h\}$)

- $\langle \begin{pmatrix} a & b\\ c & d\end{pmatrix}, S^{-1}(e)\rangle = \langle \begin{pmatrix} a & b\\ c & d\end{pmatrix}, -e\rangle = \begin{pmatrix} 0 & -1\\ 0 & 0\end{pmatrix} = \langle \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}, e\rangle = \langle S\begin{pmatrix} a & b\\ c & d\end{pmatrix}, e\rangle$
- $\langle \begin{pmatrix} a & b\\ c & d\end{pmatrix}, S^{-1}(f)\rangle = \langle \begin{pmatrix} a & b\\ c & d\end{pmatrix}, -f\rangle = \begin{pmatrix} 0 & 0\\ -1 & 0\end{pmatrix} = \langle \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}, f\rangle = \langle S\begin{pmatrix} a & b\\ c & d\end{pmatrix}, f\rangle$
- $\langle \begin{pmatrix} a & b\\ c & d\end{pmatrix}, S^{-1}(h)\rangle = \langle \begin{pmatrix} a & b\\ c & d\end{pmatrix}, -h\rangle = \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix} = \langle \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}, h\rangle = \langle S\begin{pmatrix} a & b\\ c & d\end{pmatrix}, h\rangle$

The validity of $(**)$ in general now follows from the following:

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Lemma 3: If (**) holds on the generators, then it always holds

▷

- $\langle S(h_1h_2), u\rangle = \langle S(h_2)S(h_1), u\rangle = \sum_{(u)} \langle S(h_2), u''\rangle \cdot \langle S(h_1), u'\rangle$

|| assuming validity of (**) for h_1, h_2

$$\langle h_1h_2, S^{-1}(u)\rangle = \sum_{(u)} \langle h_2, S^{-1}(u'')\rangle \cdot \langle h_1, S^{-1}(u')\rangle$$

using $\Delta\circ S^{-1} = (S^{-1}\otimes S^{-1})\circ\Delta^{op}$

Shows that if (**) holds for $h\in\{h_1,h_2\}$ then it also does for $h = h_1\cdot h_2$

- $\langle S(h), u_1u_2\rangle = \langle \Delta(S(h)), u_1\otimes u_2\rangle = \sum_{(h)} \langle S(h'')\otimes S(h'), u_1\otimes u_2\rangle = \sum_{(h)} \langle S(h''), u_1\rangle\cdot\langle S(h'), u_2\rangle$

|| validity of (**) for u_1, u_2

$$\langle h, S^{-1}(u_1u_2)\rangle = \langle h, S^{-1}(u_2)S^{-1}(u_1)\rangle = \sum_{(h)} \langle h'', S^{-1}(u_1)\rangle\cdot\langle h', S^{-1}(u_2)\rangle$$

Shows that if (**) holds for $u\in\{u_1,u_2\}$ then it also does for $u = u_1u_2$ □

This completes our proof of the Proposition. ■

Recall that $\mathbb{k}[x,y]_n = \{$degree n pol-s in $x,y\}$ has a comodule structure over $SL(2)$ (Lemma 1 of Lecture 5) $\Rightarrow$ dual space $\mathbb{k}[x,y]_n^*$ has a natural module structure over $SL(2)^*$. On the other hand, we have $U(sl_2) \xrightarrow{\psi} SL(2)^*$ from Prop 2. Therefore, we obtain $U(sl_2)$-action on $\mathbb{k}[x,y]_n^*$.

[Exercise: Verify that $\mathbb{k}[x,y]_n^*$ is an irreducible sl_2-module (hence, $\simeq V_n$)

All of today's results admit a straightforward generalization to q-case:

Exercise: a) Prove that there is an algebra homomorphism

$$\varphi: M_q(2)^{op} \longrightarrow U_q(sl_2)^* \text{ determined by } \begin{pmatrix} a & b \\ c & d\end{pmatrix} \longmapsto \rho_{L(1,+)}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u)\end{pmatrix}$$

b) Verify that the induced pairing $\langle\,,\,\rangle: M_q(2)\times U_q(sl_2)\to\mathbb{k}$ is bialgebra pairing

c) Show that $\varphi(\det_q) = 1$

d) Verify that the induced pairing $\langle\,,\,\rangle: SL_q(2)\times U_q(sl_2)\to\mathbb{k}$ is Hopf pairing

e) Prove that composing $\psi: U_q(sl_2)\to SL_q(2)^*$ with $SL_q(2)^*$-action on $\mathbb{k}_q[x,y]_n^*$ we obtain an irreducible $U_q(sl_2)$-module isomorphic to $L(n,+)$.