Lecture #17

Today & next week: Quantum groups $U_q(\mathfrak{g})$ associated to simple Lie algebras $\mathfrak{g}$.

Recall:

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{simple Lie algebras}\\ \text{of finite dimension}\end{array}\right\} & \xleftrightarrow{1:1} & \left\{\begin{array}{c}\text{Dynkin diagrams}\\ A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\end{array}\right\} \\
\Big\updownarrow 1:1 & & \Big\updownarrow 1:1 \\
\left\{\begin{array}{c}\text{reduced indecomposable}\\ \text{root systems}\end{array}\right\} & & \left\{\begin{array}{c}\text{Cartan matrices}\\ \text{(indecomposable}\end{array}\right\}
\end{array}$$

Here: A square matrix $A=(a_{ij})_{i,j\in I}$ is an indecomposable Cartan matrix if:

- $a_{ii}=2$
- $a_{ij}\in\mathbb{Z}_{\le 0}$ $\forall i\ne j$ & $a_{ij}=0 \Leftrightarrow a_{ji}$
- $\nexists$ $\emptyset\ne J\subsetneq I$ s.t. $a_{ij}=0$ $\forall i\in I\setminus J, j\in J$ ($\leftarrow$ this is what indecomposable means)
- A is diagonalizable i.e. $\exists$ $D=\begin{pmatrix} d_1 & & 0\\ & d_2 & \\ 0 & & \ddots\end{pmatrix}$ s.t. $\begin{cases} D\cdot A\text{ - symmetric} \\ d_i\in\mathbb{R}_{>0}\ \forall i\in I\end{cases}$, i.e. $d_ia_{ij}=d_ja_{ji}$
- Most importantly, the quadratic form defined by symmetric matrix is positive definite.

Fact: a) Such indecomposable Cartan matrices are classified by above Dynkin diagrams

b) One can choose D so that $d_i\in\{1,d\}$ $\forall i\in I$ with $d=1,2,$ or 3.

c) Starting from such a matrix $A=(a_{ij})$ one recovers a simple finite-dimen. Lie algebra $\mathfrak{g}(A)$ via the Chevalley-Serre theorem. In other words, $\mathfrak{g}(A)$ is defined by generators & relations:

Generators: $\{e_i, f_i, h_i\}_{i\in I}$

Relations:
$$\begin{cases}
[h_i,h_j]=0 \\
[h_i,e_j]=a_{ij}\cdot e_j \\
[h_i,f_j]=-a_{ij}\cdot f_j \\
[e_i,f_j]=\delta_{ij}\cdot h_i \\
\underbrace{[e_i,[e_i,\ldots[e_i}_{1-a_{ij}},e_j]\ldots]]=0 \quad \forall i\ne j \\
\underbrace{[f_i,[f_i,\ldots[f_i}_{1-a_{ij}},f_j]\ldots]]=0 \quad \forall i\ne j
\end{cases}$$
"Serre relations"

d) An important role in Lie theory of simple Lie algebras is played by

- Root lattice $Q:=\bigoplus_{i\in I}\mathbb{Z}\alpha_i$ "simple root" & $\boxed{\text{bilinear form } (\,,\,) \text{ on } Q\otimes_{\mathbb{Z}}\mathbb{R} \text{ via } (\alpha_i,\alpha_j)=d_ia_{ij}=d_ja_{ji}}$
- Weight lattice $P:=\bigoplus_{i\in I}\mathbb{Z}\varpi_i$ "fundamental weight", where $(\varpi_i,\alpha_j)=\delta_{ij}\cdot\frac{(\alpha_j,\alpha_j)}{2}$

We shall now define $U_q(\mathfrak{g})$ that is constructed to be a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$ (same way as we had $\mathfrak{g}=\mathfrak{sl}_2$ case before).

Def: The Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbb{k}$-algebra generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i\in I}$ subject to the following relations:

$$\begin{cases} K_iK_j=K_jK_i, \quad K_i^{\pm1}K_i^{\mp1}=1 \\ K_iE_j=q_i^{a_{ij}}E_jK_i \\ K_iF_j=q_i^{-a_{ij}}F_jK_i \\ [E_i,F_j]=\delta_{ij}\cdot\dfrac{K_i-K_i^{-1}}{q_i-q_i^{-1}} \\ \left.\begin{array}{l}\displaystyle\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_{q_i}E_i^{1-a_{ij}-r}E_jE_i^r=0 \\ \displaystyle\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_{q_i}F_i^{1-a_{ij}-r}F_jF_i^r=0\end{array}\right\} \text{"}q\text{-Serre" relations}\end{cases}$$

where $q_i:=q^{d_i}$ with d_i as above, i.e. $d_i=\frac{(\alpha_i,\alpha_i)}{2}$.

As an intermediate (simpler but much larger) object, we shall often need:

Def: Let $\bar{U}_q(\mathfrak{g})$ be an algebra defined in the same way as $U_q(\mathfrak{g})$, but without imposing the last two "q-Serre relations".

Thus, we have $\bar{U}_q(\mathfrak{g})\overset{\pi}{\twoheadrightarrow}U_q(\mathfrak{g})$ with the kernel $\mathrm{Ker}(\pi)$ generated by:

$$\boxed{\begin{array}{ll} u_{ij}^+:=\displaystyle\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_{q_i}E_i^{1-a_{ij}-r}E_jE_i^r=0 & \\ & \text{for } i\neq j \\ u_{ij}^-:=\displaystyle\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_{q_i}F_i^{1-a_{ij}-r}F_jF_i^r=0 & \end{array}}$$

Def. a) Let U_q^-, U_q^0, U_q^+ denote the subalgebras of $U_q(\mathfrak{g})$ generated by $\{F_i\}_{i\in I}$, $\{K_i^{\pm1}\}_{i\in I}$, and $\{E_i\}_{i\in I}$, respectively.

b) Let $\bar{U}_q^-, \bar{U}_q^0, \bar{U}_q^+$ be the analogous subalgebras of $\bar{U}_q(\mathfrak{g})$

c) Evoking the root lattice Q, we set $K_{\alpha=\sum_{i\in I}n_i\alpha_i}:=\prod_{i\in I}K_i^{n_i}$ $\forall n_i\in\mathbb{Z}$.

Not only $U_q(\mathfrak{g})$ generalizes $U_q(\mathfrak{sl}_2)$, but vice versa $U_q(\mathfrak{sl}_2)$-theory provides a powerful tool to study $U_q(\mathfrak{g})$-theory (similar to Lie algebras setup) through the following result:

Lemma 1: For every $i\in I$, we have an algebra homomorphism
$$\iota_{\alpha_i}: U_{q_i}(\mathfrak{sl}_2) \longrightarrow U_q(\mathfrak{g}) \text{ via } E\mapsto E_i,\ F\mapsto F_i,\ K^{\pm1}\mapsto K_i^{\pm1}$$

▸ Obvious by comparing the defining relations ∎

Same is true for $U_{q_i}(\mathfrak{sl}_2)\longrightarrow \overline{U}_q(\mathfrak{g})$, since there are no Serre rel-s in $U_{q_i}(\mathfrak{sl}_2)$

In particular, we obtain the following commutation f-las:
$$\boxed{\begin{aligned}&[E_i, F_i^r] = [r]_{q_i}\cdot F_i^{r-1}\cdot [K_i; 1-r]_{q_i}\\ &[F_i, E_i^r] = -[r]_{q_i}\cdot E_i^{r-1}\cdot [K_i; r-1]_{q_i}\end{aligned}}$$

To simplify many proofs later on (especially when establishing counterparts of identities with E_i's replaced by F_i's), we note the following:

Lemma 2: a) Both $U_q(\mathfrak{g})$ and $\overline{U}_q(\mathfrak{g})$ admit unique algebra automorphisms ω, called "Cartan involution", which are determined by
$$\boxed{\omega: E_i\longmapsto F_i,\ F_i\longmapsto E_i,\ K_i^{\pm1}\longmapsto K_i^{\mp1}}$$

b) Both $U_q(\mathfrak{g})$ and $\overline{U}_q(\mathfrak{g})$ admit unique algebra antiautomorphisms σ, determined by
$$\boxed{\sigma: E_i\longmapsto E_i,\ F_i\longmapsto F_i,\ K_i^{\pm1}\longmapsto K_i^{\mp1}}$$

Exercise (easy): Check this!

Similarly to the $\mathbb{Z}$-grading we considered on $U_q(\mathfrak{sl}_2)$, we have the following:

Lemma 3: The algebras $U_q(\mathfrak{g})$ and $\overline{U}_q(\mathfrak{g})$ are Q-graded algebras via (Q — root lattice)
$$\deg(E_i)=\alpha_i,\ \deg(F_i)=-\alpha_i,\ \deg(K_i^{\pm1})=0.$$
Moreover $\forall \alpha,\mu\in Q$, we have
$$\deg(x)=\mu \Rightarrow K_\alpha x K_\alpha^{-1} = q^{(\alpha,\mu)}\cdot x$$

Exercise (easy): Check this!

Our first result equips $U_q(\mathfrak{g})$ with a Hopf algebra structure:

Theorem 1: There is a unique Hopf algebra structure on $U_q(\mathfrak{g})$ with the coproduct Δ, counit ε, and antipode S determined by:

$$\Delta:\ E_i \mapsto E_i\otimes 1 + K_i\otimes E_i,\quad F_i\mapsto F_i\otimes K_i^{-1} + 1\otimes F_i,\quad K_i^{\pm1}\mapsto K_i^{\pm1}\otimes K_i^{\pm1}$$

$$\varepsilon:\ E_i\mapsto 0,\quad F_i\mapsto 0,\quad K_i^{\pm1}\mapsto 1$$

$$S:\ E_i\mapsto -K_i^{-1}E_i,\quad F_i\mapsto -F_iK_i,\quad K_i^{\pm1}\mapsto K_i^{\mp1}$$

We start by establishing a similar result for $\bar{U}_q(\mathfrak{g})$

Proposition 1: There is a unique Hopf algebra structure on $\bar{U}_q(\mathfrak{g})$ determined on the generators by the above f-las

▶ First, one needs to check the above assignments indeed give rise to algebra homomorphisms, i.e. each of them is compatible with the defining rel-s of $\bar{U}_q(\mathfrak{g})$. The first 3 rel-s are easy (exercise!). For the 4^{th} rel-n on $[E_i,F_j]$, it follows from $\mathfrak{sl}_2$-case for $i=j$ (see Lemma 1). Hence, it only remains to check compatibility with $[E_i,F_j]=0\ \forall i\neq j$.

- $[\Delta(E_i),\Delta(F_j)] = [E_i\otimes 1 + K_i\otimes E_i,\ 1\otimes F_j + F_j\otimes K_j^{-1}] = K_i\otimes \underbrace{[E_i,F_j]}_{=0} + \underbrace{[E_i,F_j]}_{=0}\otimes K_j^{-1} + [K_i\otimes E_i, F_j\otimes K_j^{-1}] = K_iF_j\otimes E_iK_j^{-1} - F_jK_i\otimes K_j^{-1}E_i = K_iF_j\otimes E_iK_j^{-1}\underbrace{(1-q^{d_ia_{ij}-d_ja_{ji}})}_{=0 \text{ as } d_ia_{ij}=d_ja_{ji}}$ ✓
- $[S(E_i),S(F_j)] = [-K_i^{-1}E_i, -F_jK_j] = K_i^{-1}E_iF_jK_j - F_jK_jK_i^{-1}E_i$
$$= E_iF_jK_i^{-1}K_j\cdot q^{d_ia_{ij}-2d_i} - F_jE_iK_jK_i^{-1}\cdot q^{d_ja_{ji}-2d_i} = \underbrace{[E_i,F_j]}_{=0}\cdot K_i^{-1}K_j\cdot q^{d_ia_{ij}-2d_i}$$
- $[\varepsilon(E_i),\varepsilon(F_j)]=0$

Second, we need to check thus defined Δ,ε,S satisfy all needed compatibilities. This is a direct check (exercise!) Note that actually S is recovered immediately from f-las for Δ,ε

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The following technical computation is left for Homework Assignment:

Lemma 4: For $i\neq j$, verify the following f-las for coproduct of $u_{ij}^{\pm}\in\overline{U}_q(\mathfrak{g})$:
$$\Delta(u_{ij}^+)=u_{ij}^+\otimes 1+K_i^{1-a_{ij}}K_j\otimes u_{ij}^+\ ,\qquad \Delta(u_{ij}^-)=1\otimes u_{ij}^-+u_{ij}^-\otimes K_i^{-1+a_{ij}}K_j^{-1}.$$

[Exercise: Prove this

Combining Lemma 4 and $\underset{\text{product}}{\mu}\circ(\overset{\text{antipode}}{S}\otimes id)\circ\underset{\text{coproduct}}{\Delta}=\overset{\text{unit}}{\eta}\underset{\text{counit}}{\varepsilon}$, we immediately get:

Lemma 5: $S(u_{ij}^+)=-K_i^{-1+a_{ij}}K_j^{-1}\cdot u_{ij}^+\ ,\quad S(u_{ij}^-)=-u_{ij}^-K_i^{1-a_{ij}}K_j$

Proof of Theorem 1

According to Lemmas 4-5, we get (where $\pi:\overline{U}_q(\mathfrak{g})\to U_q(\mathfrak{g})$)
$$S(\mathrm{Ker}\,\pi)\subseteq \mathrm{Ker}\,\pi$$
$$\Delta(\mathrm{Ker}\,\pi)\subseteq \mathrm{Ker}\,\pi\otimes\overline{U}_q(\mathfrak{g})+\overline{U}_q(\mathfrak{g})\otimes \mathrm{Ker}\,\pi.$$
We also clearly have $\varepsilon(\mathrm{Ker}\,\pi)=0$. Thus, Proposition 1 $\Rightarrow$ Theorem 1 ∎

Our next result will establish the triangular decomposition of $U_q(\mathfrak{g})$.
We start this with the following result on $\overline{U}_q(\mathfrak{g})$ – to be proved next time

Theorem 2: The elements $\left\{F_{i_1}\dots F_{i_k}\cdot\prod_{i\in I}K_i^{n_i}\cdot E_{j_1}\dots E_{j_l}\ \middle|\ \begin{matrix}k\geq 0,\ l\geq 0,\ n_i\in\mathbb{Z}\\ i_a, j_b\in I\end{matrix}\right\}$ form a basis of the algebra $\overline{U}_q(\mathfrak{g})$