Lecture #18

Our first goal is to prove the result stated in the very end of Lecture #17.

Theorem 1: The elements $\{F_{i_1}\dots F_{i_k}\cdot\prod_{i\in I}K_i^{n_i}\cdot E_{j_1}\dots E_{j_l}\}$ form a basis of $\bar{U}_q(\mathfrak{g})$

We start with the following few lemmas, but first let's set notations.

For any sequence of simple roots $J=(j_1,\dots,j_r)\in I^r \leftrightarrow (\alpha_{j_1},\dots,\alpha_{j_r})\in\Pi^r$, set:

$$E_J:=E_{j_1}\dots E_{j_r},\quad F_J:=F_{j_1}\dots F_{j_r},\quad wt(J):=\alpha_{j_1}+\dots+\alpha_{j_r}\Rightarrow K_{wt(J)}=K_{j_1}\dots K_{j_r}$$

Lemma 1: $\forall J$ as above $\exists$ Laurent polynomials $C^J_{A,B}(t)\in\mathbb{Z}[t,t^{-1}]$ whose coeff-s do not depend on q, parametrized by sequences A,B as above s.t. $wt(A)+wt(B)=wt(J)$ so that

$$\Delta(E_J)=\sum_{A,B}C^J_{A,B}(q)\cdot E_AK_{wt(B)}\otimes E_B,\quad \Delta(F_J)=\sum_{A,B}C^J_{A,B}(q^{-1})F_A\otimes K^{-1}_{wt(A)}F_B$$

[Exercise (easy): Prove this lemma by induction on the length of J.

Lemma 2 (Verma-modules for $\bar{U}_q(\mathfrak{g})$): Choose a collection of constants $c_i\in\mathbb{k}^*\ \forall i\in I$. Let M_c be the vector space with basis $\{v_J\}$, parametrized by all finite sequences J of simple roots (including $J=\emptyset$). There is a natural action $\bar{U}_q(\mathfrak{g})\curvearrowright M_c$ given by the following explicit operators:

$$F_i(v_J)=v_{(i,J)}$$

$$K_i(v_J)=c_i\cdot q_i^{-(a_{ij_1}+\dots+a_{ij_r})}\cdot v_J\qquad\text{where } J=(j_1,j_2,\dots,j_r)$$

$$E_i(v_J)=\sum_{\substack{1\le k\le r\\ j_k=i}}\frac{c_iq_i^{-(a_{i,j_{k+1}}+\dots+a_{ij_r})}-c_i^{-1}\cdot q_i^{(a_{i,j_{k+1}}+\dots+a_{i,j_r})}}{q_i-q_i^{-1}}v_{(j_1,\dots,j_{k-1},j_{k+1},\dots,j_r)}$$

▶ These formulas define endomorphisms of M_c, and clearly K_i-action is invertible. It thus remains to check the defining rel-s of $\bar{U}_q(\mathfrak{g})$ for these operators:

1) $K_iF_j=q_i^{-a_{ij}}F_jK_i$ is clearly preserved as
$K_iF_jv_J=K_iv_{(j,J)}=c_i\cdot q_i^{-a_{ij}}\cdot q_i^{-(a_{ij_1}+\dots+a_{ij_r})}v_{(j,J)}=q_i^{-a_{ij}}F_jK_iv_J$

2) $K_iE_j=q_i^{a_{ij}}E_jK_i$ is also clearly preserved as each sequence A featuring as v_A in $E_i(v_J)$ has $wt(A)=wt(J)-\alpha_i$

3) $[E_i,F_j]=0$ for $i\neq j$ is also clear as E_i detects all el-s of J equal to i, which is not affected by adding index j first, and coeff-s don't change

Lecture #18

(Continuation)

4) Finally, checking $[E_i,F_i]=\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}$, we see that $E_iF_iv_J$ differs from $F_iE_iv_J$ at a single term - the first one:

$$[E_i,F_i]v_J = \frac{c_i q_i^{-(a_{ij_1}+\dots+a_{ij_r})} - c_i^{-1} q_i^{(a_{ij_1}+\dots+a_{ij_r})}}{q_i-q_i^{-1}}\, v_J = \frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}\, v_J$$

■

Twisting the above $\bar{U}_q(\mathfrak{g})$-module by a Cartan involution

$$\omega: E_i \mapsto F_i,\ F_i \mapsto E_i,\ K_i \mapsto K_i^{-1}$$

and relabelling $c_i^{-1} \mapsto c_i$, we immediately get:

Lemma 3 (Verma-modules for $\bar{U}_q(\mathfrak{g})$): Choose a collection of constants $c_i \in \mathbb{k}^*$ $\forall i \in I$. There is a natural $\bar{U}_q(\mathfrak{g})$-action on $M_c^- \overset{\text{v. space}}{:=} M_c$ given by the following f-las:

$$E_i(v_J) = v_{(i,J)}$$

$$K_i(v_J) = c_i \cdot q_i^{(a_{ij_1}+\dots+a_{ij_r})} \cdot v_J$$

$$F_i(v_J) = \sum_{1\le k\le r}^{j_k=i} \frac{c_i^{-1} q_i^{-(a_{ij_{k+1}}+\dots+a_{ij_r})} - c_i q_i^{(a_{ij_{k+1}}+\dots+a_{ij_r})}}{q_i-q_i^{-1}}\, v_{(j_1,\dots,j_{k-1},j_{k+1},\dots,j_r)}$$

Let $\bar{U}_q^-$ be subalgebra of $\bar{U}_q(\mathfrak{g})$ generated by $\{F_i\}_{i\in I}$

$\bar{U}_q^+$ —"— $\{E_i\}_{i\in I}$

$\bar{U}_q^0$ —"— $\{K_i^{\pm 1}\}_{i\in I}$

As the first approximation to Theorem 1, we have:

Corollary 1: $\bar{U}_q^-$ is isomorphic to a free algebre in F_i's

$\bar{U}_q^+$ —"— E_i's

► By definition $\bar{U}_q^-$ is spanned by $\{F_J\}$. It remains to show $\{F_J\}$ - lin. indep. If not, we would have a nontrivial relation $\sum_J a_J F_J = 0$ with not all a_J zero. Applying this to $v_\emptyset \in M_c$, we get $\sum_J a_J v_J = 0$. But $\{v_J\}$ - basis of M_c for any c_i's $\Rightarrow$ all $a_J = 0$ $\Rightarrow$ Contradiction! Thus, $\bar{U}_q^- \simeq \mathbb{k}\langle F_i\rangle_{i\in I}$

Likewise, using M_c^- instead of M_c, one shows that $\bar{U}_q^+ \simeq \mathbb{k}\langle E_i\rangle_{i\in I}$

■

We shall now prove Thm 1 following similar ideas, but using both M_c & M_c^-!

Lecture #18

Proof of Theorem 1

- Easy: Given elements indeed span $\bar{U}_q(\mathfrak{g})$. To this end, one can easily show that $\mathrm{span}\{F_{i_1}\dots F_{i_k}\prod K_i^{n_i}\cdot E_{j_1}\dots E_{j_l}\}$ is invariant w.r.t. left multiplication (just use the defining rel-s). As it contains 1, it must equal all $\bar{U}_q(\mathfrak{g})$.

- Harder: Lin. independence of $\{F_{J^-}K_\mu E_{J^+}\}$ where $J^\pm$ - sequences of simple roots, $\mu\in Q$

Assume not, i.e. there is a nontrivial vanishing linear combination

$$X=\sum_{J^-,\mu,J^+} a_{J^-,\mu,J^+}\cdot F_{J^-}K_\mu E_{J^+}=0 \text{ in } \bar{U}_q(\mathfrak{g})$$

Choose J_0^- s.t. $\exists\ J^+,\mu$ with $a_{J_0^-,\mu,J^+}\neq 0$ and $\mathrm{wt}(J_0^-)$ - maximal.

Let $c_i'=1\ \forall i$ and choose any $c_i\in\mathbb{k}^*\ \forall i$. We consider the action

$$\boxed{\bar{U}_q(\mathfrak{g})\curvearrowright M_{c'}\otimes M_c} \leftarrow \text{tensor product of modules from Lemmas 2\&3.}$$

Then: $0=X(v_\varnothing\otimes v_\varnothing')=\sum_{J^-,\mu,J^+} a_{J^-,\mu,J^+}F_{J^-}K_\mu E_{J^+}(v_\varnothing\otimes v_\varnothing')$

$$\underbrace{\sum_{J^-,\mu,J^+} a_{J^-,\mu,J^+}\cdot \overset{\text{multiplicative in }\mu}{c^\mu}\cdot q^{(\mu,\mathrm{wt}J)}\cdot F_{J^-}(v_\varnothing\otimes v_J)} = \sum_{J^-,\mu,J^+}{}' a_{J^-,\mu,J^+}F_{J^-}K_\mu(v_\varnothing\otimes v_{J^+}') \quad \Big\|{}^{E_i(v_\varnothing)=0\ \forall i}_{+\text{Lemma 1}}$$

(Apply Lemma 1 to evaluate $F_J(v_\varnothing\otimes v_J)$).

In particular, by the choice of J_0^-, we note that comparing coeff-s of $v_{J_0^-}\otimes v_{J_0^+}'$ (for any J_0^+) we get

$$\boxed{0=\sum_{\mu} a_{J_0^-,\mu,J_0^+}\cdot c^\mu\cdot q^{(\mu,\mathrm{wt}J_0^+)-(\mathrm{wt}J_0^-,\mathrm{wt}J_0^+)} \quad \forall c_i}$$

If $\mu=\sum n_i\alpha_i \Rightarrow c^\mu=\prod c_i^{n_i}$ by def-n. Hence, above pol-l vanishes (if $|\mathbb{k}|<\infty$, extend the field!) Therefore, $a_{J_0^-,\mu,J_0^+}\cdot q^{\cdots}=0\ \forall J_0^-,\mu,J_0^+$.

Contradiction with the choice of J_0^-. Thus, $\{F_{J^-}K_\mu E_{J^+}\}$ - basis of $\bar{U}_q(\mathfrak{g})$.