Lecture # 19

Last time we proved:

Theorem: The elements $\{F_{i_1}\cdots F_{i_k}\cdot\prod_i K_i^{n_i}\cdot E_{j_1}\cdots E_{j_\ell}\}$ form a basis of $\tilde{U}_q(\mathfrak{g})$

As an important corollary, we obtain the following triangular decomposition for $\tilde{U}_q(\mathfrak{g})$:

Corollary: a) The multiplication map

$$\tilde{U}_q^-\otimes\tilde{U}_q^0\otimes\tilde{U}_q^+\xrightarrow[\text{mult}]{}\tilde{U}_q(\mathfrak{g})$$

is an isomorphism of vector spaces.

b) The algebra $\tilde{U}_q^-$ is isomorphic to the free algebra generated by F_i.

c) The algebra $\tilde{U}_q^+$ is isomorphic to the free algebra generated by E_i.

d) The algebra $\tilde{U}_q^0$ is isomorphic to the algebra of Laurent polynomials in K_i.

However, the key actor of our study is the quantum group $U_q(\mathfrak{g})$, and we wish to have similar results for it. Constructing a basis is a rather nontrivial task and shall be done later in the course.

Today: triangular decomposition for $U_q(\mathfrak{g})$

To this end, we recall that $\mathrm{Ker}(\tilde{U}_q(\mathfrak{g})\twoheadrightarrow U_q(\mathfrak{g}))$ is a 2-sided ideal gen-d by q-Serre relations. The key tool for today will be the interpretation of the latter through the adjoint action

Def: Given any Hopf algebra H, it admits a so-called adjoint action on itself

$$\boxed{H\overset{ad}{\curvearrowright}H\quad\text{via}\quad ad(x)(y)=\sum_{(x)}x'\cdot y\cdot S(x'')}$$

Exercise (mandatory): Verify this is indeed an action.

Remark: a) For $H=\mathbb{C}[G]$, we recover $ad(g)(h)=ghg^{-1}\quad\forall g\in G$

b) For $H=U\mathfrak{g}$, we recover $ad(x)(y)=xy-yx\quad\forall x,y\in\mathfrak{g}$

We start with a few computational lemmas on the explicit f-las of $ad:\tilde{U}_q(\mathfrak{g})\curvearrowright\tilde{U}_q(\mathfrak{g})$

$ad: U_q(\mathfrak{g})\curvearrowright U_q(\mathfrak{g})$

Lemma 1: For any $i\in I$ and $x\in \overline{U}_q(\mathfrak{g})$ or $U_q(\mathfrak{g})$, we have:

$$ad(E_i)x = E_i x - K_i\cdot x\cdot K_i^{-1}E_i,\quad ad(F_i)x=(F_i x - xF_i)K_i=[F_i,x]K_i,\quad ad(K_i)x = K_i\cdot x\cdot K_i^{-1}$$

[Exercise (easy): prove above

Evoking the explicit formulas for $\Delta(E^i)$, $\Delta(F^i)$, $S(E^i)$, $S(F^i)$ in $U_q(\mathfrak{sl}_2)$ as well as algebra (injective) homomorphisms $\begin{cases} \iota_i: U_{q_i}(\mathfrak{sl}_2)\to \overline{U}_q(\mathfrak{g}) \\ \iota_i: U_{q_i}(\mathfrak{sl}_2)\to U_q(\mathfrak{g})\end{cases}$, we get

Lemma 2: For any $i\in I$, $n\in\mathbb{N}$, and $x\in\overline{U}_q(\mathfrak{g})$ or $U_q(\mathfrak{g})$, we have:

a) $ad(E_i^n)x = \sum_{r=0}^{n} q_i^{r(n-1)}\begin{bmatrix} n\\ r\end{bmatrix}_{q_i}\cdot(-1)^r\cdot E_i^{n-r}K_i^r\, x\, K_i^{-r}E_i^r$

b) $ad(F_i^n)x = \sum_{r=0}^{n}(-1)^{n-r}q_i^{-(n-r)(n-1)}\begin{bmatrix} n\\ r\end{bmatrix}_{q_i} F_i^r x F_i^{n-r}K_i^r$

[Exercise (easy): prove above

For $i\neq j\in I$, recall the el-s $u_{ij}^{\pm}\in\overline{U}_q(\mathfrak{g})$, so that q-Serre reads $u_{ij}^{\pm}=0$.

The following result is a direct corollary of Lemma 2:

Lemma 3: a) $ad(E_i^{1-a_{ij}})E_j = u_{ij}^+$

b) $ad(F_i^{1-a_{ij}})(F_jK_j) = u_{ij}^-\cdot K_jK_i^{1-a_{ij}}$

[Exercise (easy): prove above

With this interpretation of $u_{ij}^{\pm}$, we are now ready to prove the triangular decomposition for $U_q(\mathfrak{g})$. We start with the following lemma:

Lemma 4: For any $i\neq j\in I$ and any $k\in I$, we have:

$$[F_k, u_{ij}^+]=0 \quad\text{and}\quad [E_k, u_{ij}^-]=0$$

► We shall only check the 1st, as the 2nd follows by applying Cartan involution.

- For $k\neq i,j$, the equality $[F_k,u_{ij}^+]=0$ immediately follows from the defining relations $[F_k,E_i]=0=[F_k,E_j]$.

It thus remains to treat $k=j$ and $k=i$ cases. The first of these is a direct computation, but the 2nd will be based on Lemma 3

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(Continuation of the proof)

• For $k=j$, we have $[F_k, E_i]=0$ while $[F_k, E_j] = -\frac{K_j - K_j^{-1}}{q_j - q_j^{-1}}$. Set $n = 1-a_{ij}$. Then:

$$[F_j, u_{ij}^+] = \sum_{r=0}^{n} (-1)^r \cdot \begin{bmatrix} n \\ r \end{bmatrix}_{q_i} \cdot E_i^{n-r} \cdot \frac{K_j^{-1} - K_j}{q_j - q_j^{-1}} \cdot E_i^r \underbrace{}_{\text{pull } K_j^{\pm 1} \text{ to the rightmost part and group separately } K_j \,\&\, K_j^{-1}}$$

$$= \Big(\sum_{r=0}^{n} (-1)^r \cdot \begin{bmatrix} n \\ r \end{bmatrix}_{q_i} \cdot q_j^{-ra_{ji}}\Big) \cdot \frac{E_i^n K_j^{-1}}{q_j - q_j^{-1}} - \Big(\sum_{r=0}^{n} (-1)^r \cdot \begin{bmatrix} n \\ r \end{bmatrix}_{q_i} \cdot q_j^{ra_{ji}}\Big) \cdot \frac{E_i^n K_j}{q_j - q_j^{-1}}$$

As $q_j^{ra_{ji}} = q^{r \cdot d_j a_{ji}} = q^{r d_i a_{ij}} = q_i^{ra_{ij}} = q_i^{r(1-n)}$, the equality $[F_j, u_{ij}^+]=0$ follows from

[Exercise: $\sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_q (q^{\pm(1-n)})^r = 0$

• For $k=i$, a direct calculation of $[F_i, u_{ij}]$ will trigger a lot of terms. So instead, we shall use Lemma 3 to significantly simplify computations.

First, we have:

$[F_i, u_{ij}^+] = ad(F_i)(u_{ij}^+) \cdot K_i^{-1}$ by Lemma 1

But, we also have

$u_{ij}^+ = ad(E_i^{1-a_{ij}})(E_j)$ by Lemma 3

Hence, we need to prove $ad(F_i)ad(E_i^{1-a_{ij}})(E_j)=0$, i.e. $ad(F_i E_i^{1-a_{ij}})(E_j)=0$.

But $ad(E_i^{1-a_{ij}} F_i)(E_j) = ad(E_i^{1-a_{ij}}) \underbrace{ad(F_i)(E_j)}_{=[F_i,E_j]K_i=0}$ by the same argument,

and $F_i E_i^{1-a_{ij}} - E_i^{1-a_{ij}} F_i = -[1-a_{ij}]_{q_i} \cdot E_i^{-a_{ij}} [K_i; -a_{ij}]$ (by a calculation in $U_{q_i}(sl_2)$)

However, $ad([K_i; -a_{ij}])(E_j) = \frac{1}{q_i - q_i^{-1}} \cdot (q_i^{a_{ij}} \cdot q_i^{-a_{ij}} - q_i^{-a_{ij}} \cdot q_i^{a_{ij}}) \cdot E_j = 0$, as needed ∎

Proposition 1: Let I^+ (resp. I^-) be the 2-sided ideal of $\tilde{U}_q^+$ (resp. $\tilde{U}_q^-$) generated by $\{u_{ij}^+ \mid i \neq j\}$ (resp. $\{u_{ij}^- \mid i \neq j\}$). Then:

a) The 2-sided ideal in $\tilde{U}_q(\mathfrak{g})$ gen-d by $\{u_{ij}^+\}$ is equal to $\tilde{I}^+ = \tilde{U}_q^- \cdot \tilde{U}_q^0 \cdot I^+$

b) The 2-sided ideal in $\tilde{U}_q(\mathfrak{g})$ gen-d by $\{u_{ij}^-\}$ is equal to $\tilde{I}^- = I^- \cdot \tilde{U}_q^0 \cdot \tilde{U}_q^+$

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▶ (Proof of Prop 1)

a) It suffices to show that $\tilde{I}^+$ is a 2-sided ideal (as clearly $u^+_{ij} \in \tilde{I}^+ \subseteq (u^+_{ij})_{i\neq j}$.

Note that $\tilde{I}^+ = \mathrm{span}\{x \cdot u^+_{ij} \cdot E_{\tilde{I}} \mid x \in \bar{U}_q(\mathfrak{g}),\ E_{\tilde{I}} = E_{i_1} \dots E_{i_k}\}$. Clearly, $\tilde{I}^+$ is a left ideal. To verify it's a right ideal, we note:

1) $x \cdot u^+_{ij} \cdot E_{\tilde{I}} \cdot E_2 = x \cdot u^+_{ij} \cdot (E_{\tilde{I}} E_i)$

2) $x \cdot u^+_{ij} \cdot E_{\tilde{I}} \cdot K_2 = q^{\cdots} \cdot x K_2 u^+_{ij} \cdot E_{\tilde{I}}$

3) $x \cdot u^+_{ij} \cdot E_{\tilde{I}} F_2 = \underbrace{x F_2 u^+_{ij} E_{\tilde{I}}}_{\in \tilde{I}^+} - x \cdot \underbrace{[F_2, u^+_{ij}]}_{=0 \text{ by Lemma 4}} E_{\tilde{I}} \underbrace{- x \cdot u^+_{ij} \cdot \underbrace{[F_2, E_{\tilde{I}}]}_{\in \bar{U}^0_q \cdot \bar{U}^+_q}}_{\in \tilde{I}^+ \text{ by above checks}}$

b) Analogous argument

Finally, we can prove the main result:

Theorem 1 (triangular decomposition for $U_q(\mathfrak{g})$)

a) The multiplication map

$$U^-_q \otimes U^0_q \times U^+_q \xrightarrow{\text{mult}} U_q(\mathfrak{g})$$

is an isomorphism of vector spaces.

b) $U^-_q \simeq \bar{U}^-_q / (u^-_{ij})_{i\neq j}$

$U^+_q \simeq \bar{U}^+_q / (u^+_{ij})_{i\neq j}$

$U^0_q \simeq \bar{U}^0_q$

We first note the following corollaries:

Corollary 1: The multiplication map $U^+_q \otimes U^0_q \otimes U^-_q \to U_q(\mathfrak{g})$ is a v.space isom.

▶ Apply Cartan involution to Thm 1a)

Corollary 2: The multiplication maps $U^0_q \otimes U^+_q \to U^{\geq}_q$ and $U^-_q \otimes U^0_q \to U^{\leq}_q$ are also v.space isomorphisms (where $U^{\geq}_q$ is gen-d by $E_i, K_i^{\pm 1}$, $U^{\leq}_q$ – by $F_i, K_i^{\pm 1}$)

Finally, noting that $I^{\pm}$ from Prop 1 are Q-graded with zero components in $\{k \cdot \alpha_i\}^{i \in I}_{k \in \mathbb{N}}$ we obtain the proof of the aforementioned result:

Corollary 3: For any $i \in I$, the algebra homom. $\iota_i : U_{q_i}(\mathfrak{sl}_2) \to U_q(\mathfrak{g})$ is injective

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(Proof of Thm 1)

By construction, $\mathrm{Ker}\,(\bar{U}_q(\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g}))$ is generated by $\{u_{ij}^{\pm}\}_{i\neq j}$, which thus equals $I^- \cdot \bar{U}_q^0 \bar{U}_q^+ + \bar{U}_q^- \cdot \bar{U}_q^0 \cdot I^+$ by Prop 1. Note that

$$\boxed{\bar{U}_q^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ \Big/ \left(I^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ + \bar{U}_q^- \otimes \bar{U}_q^0 \otimes I^+\right) \simeq \left(\bar{U}_q^-/I^-\right) \otimes \bar{U}_q^0 \otimes \left(\bar{U}_q^+/I^+\right)}$$

Moreover, by triangular decomposition for $\bar{U}_q(\mathfrak{g})$, we have:

$$\boxed{\mathrm{mult}: \bar{U}_q^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ \xrightarrow[v.sp.]{\sim} \bar{U}_q(\mathfrak{g})}$$

In particular, as $(I^- \otimes \bar{U}_q^0 \otimes \bar{U}_q^+ + \bar{U}_q^- \otimes \bar{U}_q^0 \otimes I^+) \cap (\mathbb{k} \otimes \mathbb{k} \otimes \bar{U}_q^+) = \mathbb{k} \otimes \mathbb{k} \otimes I^+$, we get

$$\boxed{\bar{U}_q^+/I^+ \xrightarrow{\sim} U_q^+}$$

Likewise, we also have:

$$\boxed{\bar{U}_q^-/I^- \xrightarrow{\sim} U_q^- \quad \text{and} \quad \bar{U}_q^0 \xrightarrow{\sim} U_q^0}$$

The above boxed f-las imply all the statements of Thm 1 at once ∎