Goal: Classify simple finite-dimensional $U_q(\mathfrak{g})$-modules.

In analogy to the classical story, our classification (+ proofs) will crucially use the similar results for $U_q(\mathfrak{sl}_2)$ established earlier

Recall: $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ - root lattice

$P = \bigoplus_{i\in I} \mathbb{Z}\varpi_i$ - weight lattice.

Given a $U_q(\mathfrak{g})$-module $M$, for any $\lambda\in P$ and any group homom $\varsigma: Q\to\{\pm1\}$ define

$$\boxed{M_{\lambda,\varsigma} := \{m\in M \mid K_\mu(m) = \varsigma(\mu)\cdot q^{(\lambda,\mu)}\cdot m \quad \forall \mu\in Q\}}$$

equivalently, we require that each $K_i$ acts as $\pm q^{?}$ for $\pm$ prescribed by $\varsigma(\alpha_i)$ and ? by $(\lambda,\alpha_i)$.

**Lemma 1**: Let $M$ be a fin. dim. $U_q(\mathfrak{g})$-module. Then:

a) $M = \bigoplus_{(\lambda,\varsigma)} M_{\lambda,\varsigma}$ (here we assume char $k\neq 2$)

b) $E_i M_{\lambda,\varsigma} \subseteq M_{\lambda+\alpha_i,\varsigma}$, $F_i M_{\lambda,\varsigma}\subseteq M_{\lambda-\alpha_i,\varsigma}$ $\forall i\in I$

c) $\{E_i, F_i\}$ act nilpotently on $M$

▸ b) is obvious

c) follows from such a result for the restriction of $M$ to $U_{q_i}(\mathfrak{sl}_2)$

a) According to $U_q(\mathfrak{sl}_2)$-computations, we know that each $K_i$ acts diagonalizably with desired eigenvalues. Moreover, $[K_i, K_j]=0$ $\forall i,j$. Hence, we can simultaneously diagonalize all $K_i$ ◼

**Corollary 1**: For any f.dim $U_q(\mathfrak{g})$-module $M$, we obtain $M = \bigoplus_{\varsigma} M^{\varsigma}$, the sum taken over all gp homom $Q\to\{\pm1\}$, $M^{\varsigma} = \bigoplus_{\lambda} M_{\lambda,\varsigma}$.

**Def**: One says that $M$ is of type $\varsigma$ if $M = M^{\varsigma}$.

We claim that understanding type 1 modules allows to understand type $\varsigma$-modules for any $\varsigma$. One way to see this is to twist any type 1-module $M$ by an automorphism $\boxed{E_i\mapsto \varsigma(\alpha_i)E_i,\ F_i\mapsto F_i,\ K_i\mapsto\varsigma(\alpha_i)K_i}$

Another way to see the above rel-n is to note a bijection

$$\boxed{\begin{array}{ccl} \{\varsigma : Q \xrightarrow{hom} \{\pm 1\}\} & \overset{1:1}{\longleftrightarrow} & \{1\text{-dim } U_q(\mathfrak{g})\text{-modules}\} \\ \varsigma & \longmapsto & L(0,\varsigma) : \ K_i \text{ acts via } \varsigma(\alpha_i) \\ & & \qquad\quad E_i, F_i \text{ act by zero} \end{array}}$$

Hence, if $M$ is a type $\mathbb{1}$ module, then $M \otimes L(0,\varsigma)$ is a type $\varsigma$ module, and vice versa, if $N$ is a type $\varsigma$ module, then $N \otimes L(0,\varsigma)$ is a type $\mathbb{1}$ module.

Henceforth, we consider only type $\mathbb{1}$ modules.

**Lemma 2**: Let $M$ be a nonzero fin. dim. $U_q(\mathfrak{g})$-module. Then:

a) $\exists \lambda \in P$, $m \in M_\lambda \setminus \{0\}$ s.t. $E_i(m) = 0 \ \forall i \in I$

b) In the setup of a), $\lambda$ must be "dominant integral" i.e. $n_i = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}_{\geq 0}$

Moreover, $F_i^{n_i+1}(m) = 0 \ \forall i$.

▷ a) Use Lemma 1a) and the fact $\#\{\lambda \mid M_\lambda \neq 0\} < \infty$ as $\dim(M) < \infty$

b) Fix $i \in I$, and restrict $M$ to $U_{q_i}(\mathfrak{sl}_2) = \langle E_i, F_i, K_i^{\pm 1} \rangle$. The result now follows from $U_q(\mathfrak{sl}_2)$-story ∎

**Def**: Fix $\lambda \in P$ and define the Verma module $M(\lambda)$ via

$$\boxed{M(\lambda) := \mathrm{Ind}_{U_q^{\geq}}^{U_q(\mathfrak{g})}(\mathbb{k}_\lambda)}, \text{ where } U_q^{\geq} \curvearrowright \mathbb{k}_\lambda \text{ via } E_i \mapsto 0, \ K_i \mapsto q^{(\lambda,\alpha_i)}$$

Explicitly, the Verma module can be realized as

$$\boxed{M(\lambda) := U_q(\mathfrak{g})/J_\lambda}, \text{ where } J_\lambda \text{ is the left ideal generated by } \{E_i, K_i - q^{(\lambda,\alpha_i)} \mid i \in I\}$$

Let $v_\lambda$ denote the image of $1 \in U_q(\mathfrak{g})$ in the latter realization of $M(\lambda)$.

Clearly: $K_i(v_\lambda) = q^{(\lambda,\alpha_i)} v_\lambda$, $E_i(v_\lambda) = 0$. The following is obvious

**Lemma 3**: If $M$ is a $U_q(\mathfrak{g})$-module, then there is a bijection

$$\{m \in M_\lambda \mid E_i(m) = 0 \ \forall i\} \overset{1:1}{\longleftrightarrow} \{U_q(\mathfrak{g})\text{-morphisms } M(\lambda) \xrightarrow{\varphi} M\}$$

$$m \longmapsto \varphi \text{ s.t. } \varphi(v_\lambda) = m$$

**Remark**: Evoking the triangular decomposition of $U_q(\mathfrak{g})$, we see

$$\begin{array}{ccl} U_q^- & \xrightarrow{\sim} & M(\lambda) \quad - \text{ isomorphism of vector spaces.} \\ \overset{\Psi}{x} & \longmapsto & x(v_\lambda) \end{array}$$

Similarly to the classical case, $M(\lambda)$ has a unique simple factor, to be denoted $L(\lambda)$. The following key result for today is a $q$-version of the classical classification of simple fin. dim. $\mathfrak{g}$-modules.

**Theorem 1**: For each $\underbrace{\text{dominant integral}}_{(\text{i.e. } n_i := \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}_{\geq 0})}$ $\lambda$, the simple $U_q(\mathfrak{g})$-module $L(\lambda)$ is finite-dimensional. Moreover, the reverse is also true.

The reverse is actually immediate from Lemma 2a). To prove the first part of Thm 1, we start with:

**Lemma 4**: If $\lambda \in P$ and $i \in I$ satisfy $n_i := \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}_{\geq 0}$, then there is a nonzero $U_q(\mathfrak{g})$-morphism

$$\begin{array}{rcl} \varphi: M(\lambda-(n_i+1)\alpha_i) & \longrightarrow & M(\lambda) \\ \overset{\Psi}{v_{\lambda-(n_i+1)\alpha_i}} & \longmapsto & \overset{\Psi}{F_i^{n_i+1}(v_\lambda)} \end{array}$$

▸ Clearly $F_i^{n_i+1}(v_\lambda) \in M(\lambda)_{\lambda-(n_i+1)\alpha_i}$. Need to check $E_j F_i^{n_i+1}(v_\lambda) = 0$.

- For $j \neq i$, this is obvious as $[E_j, F_i] = 0$ and $E_j(v_\lambda) = 0$.
- For $j = i$, this boils down to a $U_q(\mathfrak{sl}_2)$-computation as:

$$E_i F_i^{n_i+1} = F_i^{n_i+1} E_i + [n_i+1]_{q_i} \cdot F_i^{n} \cdot [K_i;-n_i] \quad \text{and} \quad E_i(v_\lambda) = 0 = [K_i;-n_i](v_\lambda)$$ ▪

For the rest of the arguments, we will need a proper replacement of nilpotent operators when the underlying v. space is not fin. dimen.

**Def**: A linear operator $\alpha: V \circlearrowleft$ is locally nilpotent if

$$\forall v \in V \ \exists N \in \mathbb{Z}_{>0} \text{ s.t. } \alpha^N(v) = 0$$

**Lemma 5**: Let $V$ be a $U_q(\mathfrak{sl}_2)$-module s.t. $V=\bigoplus_{n\in\mathbb{Z}} V_n$ with $V_n=\{v\in V \mid K(v)=q^n\cdot v\}$ and $E,F$ act loc nilpotently. Then: $\dim V<\infty$ and $\dim(V_n)=\dim(V_{-n})$ $\forall n$.

▷ The proof is by induction on $\dim V_0+\dim V_1$.
If $V\neq 0$, pick some $k\in\mathbb{Z}$ and $v\in V_k\setminus\{0\}$. As $E$ acts loc. nilpotently, $\exists N>0$ s.t. $E^N(v)=0$, $E^{N-1}(v)\neq 0$. Set $v':=E^{N-1}(v)\in V_{k+2(N-1)}$. Then, since $F$ acts loc. nilpotently, $E(v')=0$, $K(v')=q^{k+2(N-1)}v'$, we see that $v'$ generates a fin. dim. submodule $\Rightarrow U_q(\mathfrak{sl}_2)(v')\simeq L(\underbrace{k'}_{k+2(N-1)},+)$
But then the quotient module $V/U_q(\mathfrak{sl}_2)(v')$ satisfies the same condition and

$$\dim\left(V/U_q(\mathfrak{sl}_2)(v')\right)_0+\dim(\cdots)_1=\dim V_0+\dim V_1-1,$$

since $\dim(L(m,+)_0)+\dim(L(m,+)_1)=1$ $\forall m$. Hence, the induction hypothesis can be applied. This implies the induction step ■

**Lemma 6**: Fix $\lambda\in P$ and $r_i,s_i\in\mathbb{Z}_{>0}$ $\forall i\in I$. Let $J$ be the left ideal of $U_q(\mathfrak{g})$ generated by $\{E_i^{r_i}, F_i^{s_i}, K_i-q^{(\lambda,\alpha_i)}\}_{i\in I}$. Then: all $\{E_i,F_i\}_{i\in I}$ act locally nilpotently on $U_q(\mathfrak{g})/J$

[ Exercise: Prove this result
Hint: Use $q$-Serre rels for the only nontrivial check.

Now we finally ready to prove Theorem 1 (the first part of it).

# Lectures 20-21

► (Proof of Theorem 1)

Let $\lambda$ be a dominant integral weight, i.e. $n_i := \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}_{\geq 0}$ $\forall i \in I$.

According to Lemma 4, we have nonzero $U_q(\mathfrak{g})$-morphisms $\varphi_i : M(\lambda-(n_i+1)\alpha_i) \longrightarrow M(\lambda)$, $v_{\lambda-(n_i+1)\alpha_i} \longmapsto F_i^{n_i+1}(v_\lambda)$

Consider the quotient of Verma $M(\lambda)$ by the sum of images:

$$\boxed{\tilde{L}(\lambda) := M(\lambda)/\sum_{i\in I} \mathrm{Im}(\varphi_i) = M(\lambda)/\sum_i U_q(\mathfrak{g})(F_i^{n_i+1} v_\lambda)}$$

$\boxed{\text{Claim: } \dim \tilde{L}(\lambda) < \infty}$ This immediately implies $\dim L(\lambda)<\infty$ as $\tilde{L}(\lambda) \twoheadrightarrow L(\lambda)$

To prove the claim, we first note that $\tilde{L}(\lambda)$ is of the form as in Lemma 6:

$$\boxed{\tilde{L}(\lambda) = U_q(\mathfrak{g})\Big/\left(\begin{matrix}\text{left ideal}\\ \text{gen-d by}\end{matrix}\ E_i,\ K_i - q^{(\lambda,\alpha_i)},\ F_i^{n_i+1}\right)}$$

Hence, all $\{E_i, F_i\}_{i\in I}$ act locally nilpotently on $\tilde{L}(\lambda)$, due to Lemma 6.

From that point onwards, the argument mimicks the one for Lie algebras $\mathfrak{g}$.

Namely, we show that the set $\{\mu : \tilde{L}(\lambda)_\mu \neq 0\}$ is stable under all "reflections" $\{s_i\}_{i\in I}$, hence, under the group $W$ they generate, called the Weyl group of $\mathfrak{g}$

Basic results: 1) Any $W$-orbit of $\mu \in P$ contains a dominant integral weight

2) The set $\{\mu \in \lambda - Q_+ \mid \mu\text{- dominant integral}\}$ - finite

$Q_+ := \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$

To prove the aforementioned invariance of $\{\mu : \tilde{L}(\lambda)_\mu \neq 0\}$, fix such $\mu$ and $i \in I$.

Consider $V := \bigoplus_{k\in\mathbb{Z}} \tilde{L}(\lambda)_{\mu+k\alpha}$, which is a module over $U_{q_i}(\mathfrak{sl}_2) = \langle E_i, F_i, K_i^{\pm 1}\rangle$

As $E_i, F_i$ act locally nilpotently on $\tilde{L}(\lambda)$, so they do on $V$, hence $\dim(V)<\infty$ and $\dim V_m = \dim V_{-m}$ by Lemma 5 (here $V_{\pm m}$ is graded w.r.t $K_i$)

But: $\tilde{L}(\lambda)_{\mu+k\alpha} = V_{2k+\frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}}$, so that $\tilde{L}(\lambda)_\mu = V_{\frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}}$ and

$$\tilde{L}(\lambda)_{\mu-\frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i} = V_{-\frac{4(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}+\frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}} = V_{-\frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}}$$

Thus: $\tilde{L}(\lambda)_\mu \neq 0 \Rightarrow \tilde{L}(\lambda)_{s_i\mu} \neq 0$ (moreover, $\dim \tilde{L}(\lambda)_\mu = \dim \tilde{L}(\lambda)_{s_i\mu}$) ∎

**Corollary 2**: For any finite dimensional $U_q(\mathfrak{g})$-module $M$, we have

$$\dim M_\mu = \dim M_{w(\mu)} \quad \forall \text{weight } \mu,\ \forall w \in W$$