Lecture #22

Our Goal for today is to prove the following two results:

Theorem 1: If $\mathrm{char}(\mathbb{k})=0$ and q is transcendental over $\mathbb{Q}$, then $\tilde{L}(\lambda)\simeq L(\lambda)$ for any dominant integral λ. Moreover, the dimensions of the weight spaces $L(\lambda)_\mu$ are the same as for the highest weight λ simple $\mathfrak{g}$-module.

Here, we recall that $\tilde{L}(\lambda)=M(\lambda)/\sum_i \mathrm{Im}(\varphi_i: M(\lambda-(n_i+1)\alpha_i)\to M(\lambda))$ surjects on $L(\lambda)$.

Theorem 2: Under the same assumptions, any finite-dimensional $U_q(\mathfrak{g})$-module is semisimple, i.e. is $\simeq\bigoplus$ simple.

Remark: When we develop some new tools, we shall see that the above results hold for any $\mathbb{k}$ and any $q\in\mathbb{k}$ which is <u>NOT a ROOT of 1</u>

Both results shall be first established in the prototypical example of $\mathbb{k}=\mathbb{Q}(v)$ with $q=v\in\mathbb{k}$ (here, v - formal variable). We also set $A:=\mathbb{Q}[v,v^{-1}]$

Key Idea: Reduce to classical case via $v\mapsto 1$, but for this we need A-modules.

Pick a dominant integral weight λ, and let V be either $L(\lambda)$ or $\tilde{L}(\lambda)$. Let $v_\lambda\in V$ denote the image of the highest weight vector in $M(\lambda)$. Then, in both cases, we see: V - highest weight module generated by the highest weight vector v_λ. As both $\tilde{L}(\lambda)$ and $L(\lambda)$ are fin. dimensional (proved last time), we get: V is spanned by finitely many elements $\{F_{\mathcal{I}}(v_\lambda)\mid \mathcal{I}\text{ - sequence of simple roots}\}$.

Def Let $V_A:=\sum_{\mathcal{I}} A\cdot F_{\mathcal{I}}(v_\lambda)$ and $V_{\mu,A}:=\sum_{\mathrm{wt}(\mathcal{I})=\lambda-\mu} A\cdot F_{\mathcal{I}}(v_\lambda)$ (so that $V_A=\bigoplus_\mu V_{\mu,A}$)

Exercise 1: Prove that $V_A, V_{\mu,A}$ are free A-modules s.t. $V_{\mu,A}\otimes_A \overset{\mathbb{Q}(v)}{\mathbb{k}} \xrightarrow{\sim} V_\mu$, $V_A\otimes_A\mathbb{k}\xrightarrow{\sim} V$

Lemma 1: The A-module V_A is stable under all $E_i, F_i, K_i^{\pm1}, [K_i;n]$ for $i\in I$, $n\in\mathbb{Z}$.

▶ Stability under $F_i, K_i^{\pm1}$ is obvious. Stability under $[K_i;n]$ follows from $[m]_{v_i=v^{d_i}}\in A=\mathbb{Q}[v,v^{-1}]$ for any $m\in\mathbb{Z}$. Finally, the stability under E_i follows from $[E_i,F_j]=\delta_{ij}[K_i;0]$ and the stability under $[K_i;0]$ proved above ■

We shall now specialize $v \mapsto 1$. To this end consider $\mathbb{Q}$-algebra homomorphism

$$\boxed{\varphi: A = \mathbb{Q}[v, v^{-1}] \longrightarrow \mathbb{C} \quad \text{with} \quad v^{\pm 1} \mapsto 1.}$$

and define

$$\boxed{\bar{V} := V_A \otimes_A \mathbb{C}\ ,\ \ \bar{V}_\mu := V_{\mu,A} \otimes_A \mathbb{C}} \qquad (\text{so that } \bar{V} = \bigoplus_\mu \bar{V}_\mu)$$

According to Lemma 1, we have $E_i, F_i, K_i, [K_i;0] \curvearrowright V$ which thus give rise to

$$\boxed{e_i, f_i, k_i, h_i \curvearrowright \bar{V} \text{ respectively}}$$

Lemma 2: The above $\{e_i, f_i, h_i\}_{i\in I}$ give rise to $\mathfrak{g} \curvearrowright \bar{V}$, while all k_i act by $\mathrm{Id}_{\bar{V}}$.

▸ • For any μ and $w \in V_{\mu,A}$, we have $K_i(w) = v^{(\mu,\alpha_i)} \cdot w$, which implies $k_i \mapsto \mathrm{Id}_{\bar{V}}$ after applying the specialization φ.

• Likewise, $[K_i;0](w) = \left[\frac{(\mu,\alpha_i)}{d_i}\right]_{v_i = v^{d_i}} \cdot w$ for any $w \in V_{\mu,A}$ implies that

$$\boxed{h_i(\bar{w}) = \frac{(\mu,\alpha_i)}{d_i} \cdot \bar{w} \quad \forall \bar{w} \in \bar{V}_{\mu,A}} \text{ as } \varphi([m]_{v^r}) = m \quad \forall m, r \in \mathbb{Z} \quad \left(\text{recall that } d_i = \frac{(\alpha_i,\alpha_i)}{2}\right)$$

The above formula for the action of $\{h_i\}$ in $\bar{V}$ implies immediately:

1) $[h_i, h_j] = 0$ in $\mathrm{End}(\bar{V})$

2) $[h_i, e_j] = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} e_j = a_{ij} e_j$ in $\mathrm{End}(\bar{V})$

3) $[h_i, f_j] = -\frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} f_j = -a_{ij} f_j$ in $\mathrm{End}(\bar{V})$

On the other hand, the equality $[e_i, f_j] = \delta_{ij} h_i$ in $\mathrm{End}(\bar{V})$ follows from $[E_i, F_j] = \delta_{ij}[K_i;0]$, while the q-Serre relations on $\{E_i\}$ or $\{F_i\}$ imply the classical Serre rel-s on $\{e_i\}$ or $\{f_i\}$, due to $\varphi\left(\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_{v_i}\right) = \binom{1-a_{ij}}{r}$ ∎

[**Remark**: Due to Exercise 1, we see that $\dim_{\mathbb{C}} \bar{V}_\mu = \mathrm{rk}_A V_{\mu,A} = \dim_{\mathbb{Q}(v)} V_\mu$

Lemma 3: $\bar{V}$ is a simple $\mathfrak{g}$-module of highest weight λ, and $\bar{V}_\mu$ is just the weight μ subspace of $\bar{V}$

▸ Know: $\bar{V}$ is highest weight module of fin. dim. Then, by the basic result about simple Lie algebras, it must be simple! The fact that $\bar{V}_\mu = (\bar{V})_\mu$ follows from the explicit action of h_i derived above ∎

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We are now ready to prove Theorem 1

(Proof of Theorem 1)

- Case 1: Prototypical example $\mathbb{k}=\mathbb{Q}(v)\ni q=v$, where v - formal variable. Since all Lemmas 1-3 hold for either $L(\lambda)$ or $\tilde{L}(\lambda)$, we deduce that $\dim_{\mathbb{C}} \overline{\tilde{L}(\lambda)}_\mu = \dim \overline{L(\lambda)}_\mu \ \forall \mu$. Evoking the previous Remark, we conclude that $\dim_{\mathbb{k}} \tilde{L}(\lambda)_\mu = \dim_{\mathbb{k}} L(\lambda)_\mu \ \forall\mu$. But we know that

$$\underset{\bigoplus_\mu \tilde{L}(\lambda)_\mu}{\tilde{L}(\lambda)\ \overset{\shortparallel}{}} \twoheadrightarrow \underset{\bigoplus_\mu L(\lambda)_\mu}{L(\lambda)\ \overset{\shortparallel}{}}.$$

Hence: $\tilde{L}(\lambda) \xrightarrow{\sim} L(\lambda)$ and furthermore $\dim_{\mathbb{k}} L(\lambda)_\mu$ coincides with the dimension of weight μ subspace of simple $\mathfrak{g}$-module of h.weight λ.

- Case 2: General case of a char$=0$ field $\mathbb{k}$ and $q\in\mathbb{k}$ transcendental over $\mathbb{Q}$. In this case, we have $\gamma: \mathbb{Q}(v) \to \mathbb{k}$, $v \mapsto q$. We shall deduce the claim from Case 1 through $\otimes_{\mathbb{Q}(v)}\mathbb{k}$.

We shall use index $\mathbb{Q}(v)$ or $\mathbb{k}$ to distinguish b/w same objects over $\mathbb{Q}(v)$, $\mathbb{k}$. First, we recall that $\tilde{L}(\lambda)_{\mathbb{Q}(v)}$ was defined through the exact sequence

$$\bigoplus_{i\in I} M(\lambda-(n_i+1)\alpha_i)_{\mathbb{Q}(v)} \xrightarrow{\oplus\varphi_i} M(\lambda)_{\mathbb{Q}(v)} \twoheadrightarrow \tilde{L}(\lambda)_{\mathbb{Q}(v)} \to 0 \qquad \text{where } n_i=\frac{(\lambda,\alpha_i)}{d_i}$$

$\Downarrow$ apply $\otimes_{\mathbb{Q}(v)}\mathbb{k}$ which is right exact

$$\underbrace{\bigoplus_{i\in I} M(\lambda-(n_i+1)\alpha_i)_{\mathbb{Q}(v)}\otimes_{\mathbb{Q}(v)}\mathbb{k}}_{\shortparallel} \xrightarrow{\oplus\varphi_i} \underbrace{M(\lambda)_{\mathbb{Q}(v)}\otimes_{\mathbb{Q}(v)}\mathbb{k}}_{\shortparallel} \to \tilde{L}(\lambda)_{\mathbb{Q}(v)}\otimes_{\mathbb{Q}(v)}\mathbb{k} \to 0$$

$$\bigoplus_{i\in I} M(\lambda-(n_i+1)\alpha_i)_{\mathbb{k}} \xrightarrow{\oplus\varphi_i} M(\lambda)_{\mathbb{k}} \longrightarrow \tilde{L}(\lambda)_{\mathbb{Q}(v)}\otimes_{\mathbb{Q}(v)}\mathbb{k} \to 0$$

where we used $M(\mu)_{\mathbb{Q}(v)}\otimes_{\mathbb{Q}(v)}\mathbb{k} \simeq M(\mu)_{\mathbb{k}}\ \forall\mu$. But the same exact sequence is also used to define $\tilde{L}(\lambda)_{\mathbb{k}}$. Thus: $\tilde{L}(\lambda)_{\mathbb{k}} \simeq \tilde{L}(\lambda)_{\mathbb{Q}(v)}\otimes_{\mathbb{Q}(v)}\mathbb{k}$

[Exercise 2: Prove that $L(\lambda)_{\mathbb{k}} \simeq L(\lambda)_{\mathbb{Q}(v)}\otimes_{\mathbb{Q}(v)}\mathbb{k}$

Combining the above two isomorphisms, we immediately see that the special setup of Case 1 actually implies the general case. ∎

So: $\boxed{L(\lambda) \simeq M(\lambda)/\sum_i U_q(\mathfrak{g})(F_i^{n_i+1}v_\lambda)}$ with $n_i=\frac{(\lambda,\alpha_i)}{d_i}$ (exactly as in classical case)

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Let us now prove Theorem 2 using some general-type arguments.

► (Proof of Theorem 2)

It suffices to prove that any extension of two simple f.d. $U_q(\mathfrak{g})$-modules splits. That is given dominant integral weights λ, μ and a short exact sequence

$$0 \to L(\lambda) \xrightarrow{\iota} V \xrightarrow{\pi} L(\mu) \to 0$$

there is a "section" $U_q(\mathfrak{g})$-module morphism $s: L(\mu) \to V$ s.t. $\pi \circ s = \mathrm{Id}_{L(\mu)}$.

The above short exact sequence (S.E.S) induces such of weight spaces

$$0 \to L(\lambda)_\nu \to V_\nu \to L(\mu)_\nu \to 0 \quad \forall \nu.$$

Case 1: $\lambda \not> \mu$ in the sense $\lambda - \mu \notin \left(\bigoplus_i \mathbb{Z}_{\geq 0}\, \alpha_i\right) \setminus \{0\}$

In this case, $V_\mu \neq 0$ but $V_{\mu+\alpha_i} = 0 \ \forall i$. Hence, any nonzero $v \in V_\mu$ is a highest weight vector. This implies $M(\mu) \xrightarrow{\psi} V$, $v_\mu \mapsto v$. As $\dim(V) < \infty$, ψ must factor through $\tilde{L}(\mu)$, due to $\mathfrak{sl}_2$-considerations. But $\tilde{L}(\mu) \simeq L(\mu)$ by Thm 1, hence, $\psi: L(\mu) \twoheadrightarrow U_q(\mathfrak{g})(v) \subseteq V$. As $L(\mu)$ is simple, we get $U_q(\mathfrak{g})v \simeq L(\mu)$, thus providing the desired section $s: L(\mu) \to V$.

Case 2: $\lambda > \mu$, i.e. $\lambda - \mu = \sum_{i \in I} k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$ and $\sum k_i \neq 0$.

We shall use the standard trick of dualizing the original S.E.S:

$$0 \to L(\mu)^* \to V^* \to L(\lambda)^* \to 0$$

Clearly $L(\mu)^*, L(\lambda)^*$ are simple f.d. $U_q(\mathfrak{g})$-modules. Their highest weights are determined in:

Exercise 3: Let $w_0 \in W$ be the longest element, so that the lowest weight of $L(\lambda)$ is $w_0\lambda$ (e.g. for $\mathfrak{g} = \mathfrak{sl}_n$, $W = S(n)$, and $w_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$). Show that $L(\lambda)^* \simeq L(-w_0\lambda)$

Therefore, V^* fits into a S.E.S:

$$0 \to L(-w_0\mu) \to V^* \underset{s^*}{\leftrightarrows} L(-w_0\lambda) \to 0.$$

But: $\lambda > \mu \Rightarrow -w_0\lambda > -w_0\mu$, hence, above sequence splits by Case 1 that is $V^* \simeq L(\lambda)^* \oplus L(\mu)^*$ as $U_q(\mathfrak{g})$-modules.

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Dualizing once again, we obtain

$$(V^*)^* \simeq \underbrace{(L(\lambda)^*)^*}_{\simeq L(-w_0(-w_0\lambda)) \simeq L(\lambda)} \oplus \underbrace{(L(\mu)^*)^*}_{\simeq L(-w_0(-w_0\mu)) = L(\mu)}$$

The result now follows from:

Exercise 4: For any fin. dimensional $U_q(\mathfrak{g})$-module V, we have
$$V \simeq (V^*)^* \text{ as } U_q(\mathfrak{g})\text{-modules}$$

(**Warning**: As $\underset{\substack{\uparrow \\ \text{S-antipode of } U_q(\mathfrak{g})}}{S^2} \neq \mathrm{id}$, the usual vector space isomorphism $V \xrightarrow{\sim} (V^*)^*$ is not a $U_q(\mathfrak{g})$-module isomorphism. However, one can easily correct it by
$$V \longrightarrow (V^*)^*$$
$$v \longmapsto (\varphi \in V^* \longmapsto \varphi(K_? v)) \quad \text{for specific ?}$$
)

Thus, we conclude $V \simeq L(\lambda) \oplus L(\mu)$ as needed ∎

Remark (for math audience): Note that $\mathrm{Ext}^1(L(\lambda), L(\mu)) = 0$ for any dominant integral λ, μ implies that $\mathrm{Ext}^1(V, W) = 0$ for any fin. dim. $U_q(\mathfrak{g})$-modules, since each of these admits a Jordan-Hölder type filtration