Lecture #23

Goal: Describe "the upper bound" for the center $Z_q(\mathfrak{g}) = Z(U_q(\mathfrak{g}))$

(to get the full description we shall employ a bialgebra pairing $U_q^{\le} \times U_q^{\ge} \to \Bbbk$ that will be introduced next time)

Our Assumptions (as last time): $\Bbbk$ - field of char $=0$, $q \in \Bbbk$ - transcendental over $\mathbb{Q}$ — we'll say "t$\mathbb{Q}$-condition" for short

- Recall that $U_q(\mathfrak{g})$ is graded by a root lattice Q, so that the degree 0 component

$$\boxed{U_q(\mathfrak{g})_0 = \{x \in U_q(\mathfrak{g}) \mid K_i x K_i^{-1} = x \;\; \forall i \in I\} = \{x \in U_q(\mathfrak{g}) \mid K_\mu x K_\mu^{-1} = x \;\; \forall \mu \in Q\}}$$

In particular, $Z_q(\mathfrak{g}) \subseteq U_q(\mathfrak{g})_0$

- **Def**: For any integral weight λ, we shall use the same notation to denote the algebra homomorphism $U_q^0 \xrightarrow{\lambda} \Bbbk$ given by $K_\mu \longmapsto q^{(\lambda,\mu)}$

- Similarly to sl_2-case, we now recall the decomposition of $\underbrace{U_q(\mathfrak{g})_0}_{\text{degree 0 part}}$ arising from the triangular decomposition of $U_q(\mathfrak{g})$:

$$\boxed{U_q(\mathfrak{g})_0 = \underbrace{U_q^0}_{\text{Cartan part}} \oplus \bigoplus_{\nu \in Q_+ \setminus \{0\}} (U_q^-)_{-\nu} \cdot U_q^0 \cdot (U_q^+)_\nu}$$

Def: Let $\pi : U_q(\mathfrak{g})_0 \twoheadrightarrow U_q^0$ be the natural projection on the 1st summand above.

$\boxed{\textbf{Lemma 1}: \ \pi \text{ is an algebra homomorphism}}$

▶ Need to show that $\bigoplus_{\nu \in Q_+ \setminus \{0\}} (U_q^-)_{-\nu} \cdot U_q^0 \cdot (U_q^+)_\nu$ is a 2-sided ideal of $U_q(\mathfrak{g})_0$

This is clear as e.g.

$$\underset{(U_q^-)_{-\nu}}{x^-} \cdot \underset{U_q^0}{x^0} \cdot \underset{(U_q^+)_\nu}{x^+} \cdot \underset{(U_q^-)_{-\mu}}{y^-} \cdot \underset{U_q^0}{y^0} \cdot \underset{(U_q^+)_\mu}{y^+} \overset{\text{triangular decomp}}{=} \sum_j x^- (z_j^- \cdot z^0 \cdot z_j^+) \cdot y^+ = \underbrace{\sum_j (x^- z_j^-) \cdot z^0 (z_j^+ \cdot y^+)}_{\text{in the above direct } \oplus}$$

■

Let us now restrict the above projection π to the center

$$\begin{tikzcd}
Z_q(\mathfrak{g}) \arrow[r, hook] \arrow[dr, "\pi|_{Z_q(\mathfrak{g})}"'] & U_q(\mathfrak{g})_0 \arrow[d, "\pi"] \\
 & U_q^0
\end{tikzcd}$$

Lemma 2: a) If $x\in Z_q(\mathfrak{g})$, then for any weight λ elt x acts on Verma $M(\lambda)$ just as $\lambda(\pi(x))\cdot Id_{M(\lambda)}$

b) Assuming "tQ-condition", the restriction $\pi|_{Z_q(\mathfrak{g})}$ is injective.

► a) $v_\lambda\in M(\lambda)$ - highest weight vector $\Rightarrow E_i(v_\lambda)=0\ \forall i \Rightarrow x(v_\lambda)=\pi(x)(v_\lambda)$.

But $K_\mu(v_\lambda)=q^{(\lambda,\mu)}\cdot v_\lambda=\lambda(K_\mu)\cdot v_\lambda \Rightarrow \pi(x)(v_\lambda)=\lambda(\pi(x))\cdot v_\lambda$.

Finally, since v_λ generates all $M(\lambda)$ and x - central, we get the result (take any $w\in M(\lambda)$, write it as $w=y(v_\lambda)$ for some $y\in U_q(\mathfrak{g})$, then: $x(w)=xy(v_\lambda)=yx(v_\lambda)=\lambda(\pi(x))\cdot y(v_\lambda)=\lambda(\pi(x))\cdot w$)

b) Assume $x\in Z_q(\mathfrak{g})$ is s.t. $\pi(x)=0 \overset{a)}{\Longrightarrow}$ x acts by ZERO on each Verma $M(\lambda)$.

But then x acts by zero on any simple quotient $L(\lambda)$ of $M(\lambda)$.

Because any f. dim. $U_q(\mathfrak{g})$-module is $\simeq \oplus L(\lambda)$ under the "tQ-condition" we thus conclude that x acts trivially on all fin. dim. modules.

Then: $x=0$ by [Hwk 4, Problem 4] ∎

Def: For any weight λ, we shall consider algebra automorphism
$$\gamma_\lambda: U_q^0\to U_q^0 \text{ given by } K_\mu\mapsto q^{(\lambda,\mu)}K_\mu$$

Note: 1) $\gamma_\lambda\circ\gamma_\mu=\gamma_{\lambda+\mu}$

2) $\mu(\gamma_\lambda(x))=(\mu+\lambda)(x)\ \forall x\in U_q^0$

Def: The (quantum) <u>Harish-Chandra homomorphism</u> is the following:
$$\boxed{\gamma_{-\rho}\circ\pi := HC: Z_q(\mathfrak{g})\longrightarrow U_q^0}$$
with the weight ρ s.t. $(\rho,\alpha_i)=\frac{1}{2}(\alpha_i,\alpha_i)\ \forall i$

explicitly: $\rho=\sum_i \underbrace{\varpi_i}_{\text{fund. weights}}=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ (all positive roots)

The reason for the above "twist" of π by the automorphism $\gamma_{-\rho}$ is that it will yield a symmetry under the Weyl gp W, which is generated by simple reflections $\{s_i\}_{i\in I}$ acting on Q via $s_i(\lambda)=\lambda-\frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$.

Lecture #23

The action $\underbrace{W}_{\text{weyl gp}} \curvearrowright \underbrace{Q}_{\text{root lattice}}$ gives rise to $W \curvearrowright U_q^\circ$ via $w(K_\mu) = K_{w(\mu)}$.

$$\boxed{\text{Lemma 3: } HC(Z_q(\mathfrak{g})) \subseteq (U_q^\circ)^W}$$

▶ Pick $x \in Z_q(\mathfrak{g})$ and let $h := HC(x) \in U_q^\circ$, so that $\pi(x) = \gamma_\rho(h)$. Then by Lemma 2: x acts on any Verma $M(\lambda)$ via $x_{|M(\lambda)} = \lambda(\gamma_\rho(h)) \cdot Id_{M(\lambda)} = (\lambda+\rho)(h) \cdot Id_{M(\lambda)}$.

Now if $n_i := \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\geq 0}$, then last time we noted that there exists a nonzero $U_q(\mathfrak{g})$-module morphism

$$\begin{array}{ccc} M(\lambda - (n_i+1)\alpha_i) & \longrightarrow & M(\lambda) \\ \psi & & \psi \\ v_{\lambda-(n_i+1)\alpha_i} & \longmapsto & F_i^{n_i+1}(v_\lambda) \end{array}$$

Note that $n_i + 1 = \frac{2(\lambda+\rho, \alpha_i)}{(\alpha_i, \alpha_i)} \Rightarrow \lambda - (n_i+1)\alpha_i = s_i(\lambda+\rho) - \rho$.

But then x should act by same constants on the above Verma modules $\Downarrow$

$$(\lambda+\rho)(h) = (s_i(\lambda+\rho))(h) \Rightarrow (\lambda+\rho)(h - s_i(h)) = 0 \quad \forall \lambda \text{ s.t. } n_i \geq 0.$$

[Exercise: The above immediately implies $h = s_i(h)$.

As W is generated by all $\{s_i\}$, we conclude $h = w(h)\ \forall w \in W$ ■

• However, unlike the $\mathfrak{sl}_2$-case treated in Lecture 12, $HC(Z_q(\mathfrak{g})) \neq (U_q^\circ)^W$. We shall now identify one more restriction on $HC(Z_q(\mathfrak{g}))$

‖ Def: Let $U_{ev}^\circ := \bigoplus_{\mu \in 2P \cap Q} \mathbb{k} \cdot K_\mu$, and set $(U_{ev}^\circ)^W := (U_q^\circ)^W \cap U_{ev}^\circ$

Note: For $\mathfrak{g} = \mathfrak{sl}_2$, have $P = \frac{1}{2}Q \Rightarrow 2P \cap Q = Q \Rightarrow U_{ev}^\circ = U_q^\circ$ (which explains why we didn't see this new phenomena)

$$\boxed{\text{Lemma 4: } HC(Z_q(\mathfrak{g})) \subseteq (U_{ev}^\circ)^W}$$

▶ Pick any $x \in Z_q(\mathfrak{g})$. By Lemma 3: $HC(x) = \sum_{\mu \in Q} a_\mu K_\mu$ s.t. $a_\mu = a_{w\mu}\ \forall\, {}^{w \in W}_{\mu \in Q}$.

Let's now consider algebra automorphisms $\tilde{\sigma}: U_q(\mathfrak{g}) \circlearrowleft$ associated to any group homomorphism $\sigma: Q \to \{\pm 1\}$ (which is the same as specifying $\sigma(\alpha_i) \in \pm 1\ \forall i$)

$$\boxed{\tilde{\sigma}: E_i \mapsto \sigma(\alpha_i) E_i, \quad F_i \mapsto F_i, \quad K_i^{\pm 1} \mapsto \sigma(\alpha_i) \cdot K_i^{\pm 1}.}$$

As $\tilde{\sigma}$ commutes with $\pi, \{\gamma_\lambda\}$, so it does with HC, i.e.

$$HC(\tilde{\sigma}(x)) = \tilde{\sigma}(HC(x))$$

Lecture #23

▶ (Continuation)

$$\left.\begin{array}{l} \text{But } \tilde{\sigma}(HC(x)) = \sum_\mu a_\mu \cdot \sigma(\mu) \cdot K_\mu \\ \qquad \| \\ \quad HC(\tilde{\sigma}(x)) \subseteq (U_q^0)^W \end{array}\right\} \Rightarrow a_\mu \cdot \sigma(\mu) = a_{w\mu} \cdot \sigma(w\mu)$$

$$\Downarrow \text{ as } a_\mu = a_{w\mu}$$

$$a_\mu(\sigma(\mu) - \sigma(w\mu)) = 0.$$

So: If $a_\mu \neq 0 \Rightarrow \sigma(\mu - w\mu) = 1 \quad \forall \sigma$.

Let's take $w = s_i$ - simple reflection, so that $\mu - s_i\mu = \frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. Then:

$$\underbrace{\sigma(\mu - s_i\mu)}_{\overset{\|}{1} \text{ by above}} = (\underbrace{\sigma(\alpha_i)}_{\in \pm 1})^{\frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}} \Rightarrow \frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)} \in 2\mathbb{Z} \Rightarrow \mu \in 2P \cap Q$$

Thus, we concluded that if $a_\mu \neq 0 \Rightarrow \mu \in 2P \cap Q$, which completes the proof ∎

Conclusion: $HC : Z_q(\mathfrak{g}) \hookrightarrow (U_{ev}^0)^W$

The proof of the following result shall require some new machinery:

Theorem 1: Assuming "+Q-conditions", $HC : Z_q(\mathfrak{g}) \xrightarrow{\sim} (U_{ev}^0)^W$ is an algebra isom