Lecture #24

Goal: Construct a bialgebra pairing $U_q^{\geq} \times U_q^{\leq} \to k$
(we shall ultimately use it to prove Theorem 1 from the end of Lecture 23).

• For any $i \in I$, consider $f_i \in (U_q^{\geq})^*$ defined via

$$\boxed{f_i(E_{\mathcal{I}} K_\mu) = \begin{cases} \frac{-1}{q_i^{-1}-q_i} & \text{if } \mathcal{I}=(\alpha_i) \\ 0, & \text{otherwise.} \end{cases}}$$

Note: 1) According to triangular decomposition $U_q^{\geq}$ is spanned by $E_{\mathcal{I}} K_\mu$.

2) Even though $E_{\mathcal{I}} K_\mu$ are linearly dependent the above def-n is well-defined. Indeed, $\mathrm{Ker}(\tilde{U}_q^{\geq} \twoheadrightarrow U_q^{\geq})$ is zero in degree α_i, while in all other degrees we set f_i to vanish.

3) One cannot define $f_{\mathcal{I}}$ for general $\mathcal{I}$ in the above fashion exactly b/c $\{E_{\mathcal{I}} K_\mu\}$ is not a basis.

• Instead, for any sequence $\mathcal{I}=(i_1,\ldots,i_r)$ of simple roots we shall define

$$\boxed{f_{\mathcal{I}} := f_{i_1} \cdot \ldots \cdot f_{i_r} \in (U_q^{\geq})^*}$$

Recall: The product on $(U_q^{\geq})^*$ is dual to coproduct on $U_q^{\geq}$.

Lemma 1: a) $f_{\mathcal{J}}(E_{\mathcal{I}} K_\mu) = f_{\mathcal{J}}(E_{\mathcal{I}})$

b) $f_{\mathcal{J}}(E_{\mathcal{I}}) = 0$ if $\mathrm{wt}(\mathcal{I}) \neq \mathrm{wt}(\mathcal{J})$

Recall: $\mathrm{wt}(i_1,\ldots,i_r) := \alpha_{i_1}+\ldots+\alpha_{i_r}$, $\in Q$ = root lattice

[Exercise: prove this lemma.

• For any integral weight $\lambda \in P$, we consider $k_\lambda \in (U_q^{\geq})^*$ defined via

$$\boxed{k_\lambda(E_{\mathcal{I}} K_\mu) = \begin{cases} q^{-(\lambda,\mu)} & \text{if } \mathcal{I}=\emptyset \\ 0 & \text{otherwise} \end{cases}}$$

Similarly to comment on f_i, the above is well-defined.

Lemma 2: a) $k_\lambda k_{\lambda'} = k_{\lambda+\lambda'}$ $\forall \lambda,\lambda'$. In particular, $[k_\lambda, k_{\lambda'}]=0$.

b) $k_\lambda f_{\mathcal{J}} = q^{-(\lambda,\mathrm{wt}\mathcal{J})} \cdot f_{\mathcal{J}} k_\lambda$

We shall present a straightforward proof below (key: product in $(U_q^{\geq})^*$ is dual to coproduct in $U_q^{\geq}$).

Lecture #24

► (Proof of Lemma 2)

Recall that $\Delta(K_\mu) = K_\mu \otimes K_\mu$ and $\Delta(E_{\mathcal{I}}) = \sum_{\substack{A,B \\ wt(A)+wt(B)=wt(\mathcal{I})}} c^{\mathcal{I}}_{A,B} \cdot E_A K_{wt(B)} \otimes E_B$

$$\Downarrow$$

$$\Delta(E_{\mathcal{I}} K_\mu) = \sum_{A,B}^{wtA+wtB=wt\mathcal{I}} {}^{\prime}\; E_A K_{wt(B)+\mu} \otimes E_B K_\mu .$$

Hence, for any $\psi \in (U_q^{\geq})^*$, we have:

$$(k_\lambda \cdot \psi)(E_{\mathcal{I}} K_\mu) = (k_\lambda \otimes \psi)(\Delta(E_{\mathcal{I}} K_\mu)) = k_\lambda(K_{wt(\mathcal{I})+\mu}) \cdot \psi(E_{\mathcal{I}} K_\mu)$$

here we used $k_\lambda(E_A K_?) = 0$ if $A \neq \emptyset$ and $c^{\mathcal{I}}_{\emptyset,\mathcal{I}} = 1$.

- For $\psi = k_{\lambda'}$, we get $(k_\lambda \cdot k_{\lambda'})(E_{\mathcal{I}} K_\mu) = \begin{cases} 0 & \text{if } \mathcal{I} \neq \emptyset \\ q^{-(\lambda+\lambda',\mu)} & \text{if } \mathcal{I} = \emptyset \end{cases} = k_{\lambda+\lambda'}(E_{\mathcal{I}} K_\mu)$. ✓

- For $\psi = f_i$, we get $(k_\lambda \cdot f_i)(E_{\mathcal{I}} K_\mu) = \begin{cases} q^{-(\lambda,\alpha_i)-(\lambda,\mu)} \cdot \frac{1}{q_i^{-1}-q_i} & , \text{if } \mathcal{I} = (i) \\ 0 & , \text{otherwise.} \end{cases}$

  But similarly: $(f_i \cdot k_\lambda)(E_{\mathcal{I}} K_\mu) = \sum_{A,B} c^{\mathcal{I}}_{A,B} f_i(E_A K_{wt(\mathcal{I})+\mu}) \cdot \underbrace{k_\lambda(E_B K_\mu)}_{\text{zero unless } B=\emptyset}$

  $$= f_i(E_{\mathcal{I}} K_{wt(\mathcal{I})+\mu}) \cdot k_\lambda(K_\mu) = \begin{cases} q^{-(\lambda,\mu)} \cdot \frac{1}{q_i^{-1}-q_i} & , \text{if } \mathcal{I} = (i) \\ 0 & , \text{otherwise.} \end{cases}$$

  So: $k_\lambda f_i = q^{-(\lambda,\alpha_i)} f_i k_\lambda$

- For $\psi = f_{\mathcal{I}} = f_{i_1} \cdots f_{i_r}$, apply the previous equality r times to get

  $$k_\lambda f_{\mathcal{I}} = f_{\mathcal{I}} k_\lambda \cdot q^{-(\lambda,\alpha_{i_1}) - \dots - (\lambda,\alpha_{i_r})}$$

$\blacksquare$

Evoking the triangular decomposition & basis for $U_q^{\leq}$, we define the linear map

$$\boxed{\varphi: U_q^{\leq} \longrightarrow (U_q^{\geq})^* \quad \text{via} \quad F_{\mathcal{I}} K_\mu \longmapsto f_{\mathcal{I}} k_\mu .}$$

Lemma 3: φ - algebra homomorphism

► As $F_{\mathcal{I}} K_\mu F_{\mathcal{J}} K_\nu = F_{(\mathcal{I},\mathcal{J})} K_{\mu+\nu} \cdot q^{-(\mu, wt\mathcal{J})}$
$f_{\mathcal{I}} k_\mu f_{\mathcal{J}} k_\nu = f_{(\mathcal{I},\mathcal{J})} k_{\mu+\nu} \cdot q^{-(\mu, wt\mathcal{J})}$ by Lemma 2 $\Bigg\} \Rightarrow$ claim follows

$\blacksquare$

Lecture #24

- Define a bilinear pairing

$$\boxed{(\ ,\ ): \bar{U}_q^{\leq} \times U_q^{\geq} \longrightarrow \Bbbk \quad \text{via} \quad (y,x) = \varphi(y)(x)}$$

Note: 1) $(y,x) = 0$ for $y \in (\bar{U}_q^-)_{-\nu}$, $x \in (U_q^+)_\mu$ with $\mu \neq \nu$

2) $(yK_\lambda, xK_\nu) = q^{-(\lambda,\mu)} \cdot (y,x)$ $\forall\ y \in \bar{U}_q^-$, $x \in U_q^+$, and $\lambda, \mu \in Q$

Lemma 4: a) $(y_1 y_2, x) = (y_1 \otimes y_2, \Delta(x))$ $\quad \forall\ y_1, y_2 \in \bar{U}_q^{\leq}$, $x \in U_q^{\geq}$

b) $(y, x_1 x_2) = (\Delta(y), x_2 \otimes x_1)$ $\quad \forall\ y \in \bar{U}_q^{\leq}$, $x_1, x_2 \in U_q^{\geq}$

▶ a) Obvious:

$$(y_1 y_2, x) = \varphi(y_1 y_2)(x) = (\varphi(y_1)\varphi(y_2))(x) = (\varphi(y_1) \otimes \varphi(y_2))(\Delta(x)) = (y_1 \otimes y_2, \Delta(x))$$

b) Suffices to check for y-generator, i.e. $y = K_i$ or $y = F_i$.

- $y = K_i$

If $x_1 \notin U_q^0$ or $x_2 \notin U_q^0$, then $(K_i, x_1 x_2) = 0 = (K_i \otimes K_i, x_2 \otimes x_1)$

If $x_1 = K_\mu$, $x_2 = K_\nu$, then $(K_i, x_1 x_2) = q^{-(\alpha_i, \mu+\nu)} = (K_i \otimes K_i, x_2 \otimes x_1)$

- $y = F_i$

If $wt(x_1) + wt(x_2) \neq \alpha_i$, then both sides are zero. Otherwise, one of two occurs:

1) $x_1 = E_i K_\mu$, $x_2 = K_\nu$

$(F_i, E_i K_\mu K_\nu) = \frac{1}{q_i^{-1} - q_i}$ ✓

$(\Delta(F_i), K_\nu \otimes E_i K_\mu) = (F_i, E_i K_\mu) = 1/q_i^{-1} - q_i$

2) $x_1 = K_\nu$, $x_2 = E_i K_\mu$

$(F_i, K_\nu E_i K_\mu) = q^{(\alpha_i, \nu)} \cdot \frac{1}{q_i^{-1} - q_i}$ ✓

$(\Delta(F_i), E_i K_\mu \otimes K_\nu) = (F_i, E_i K_\mu) \cdot (K_i^{-1}, K_\nu) = q^{(\alpha_i, \nu)} \cdot \frac{1}{q_i^{-1} - q_i}$

Remark: Using our notations of Lecture 15, the above lemma means that the pairing

$$U_q^{\geq} \times \bar{U}_q^{\leq} \longrightarrow \Bbbk$$
$$x, y \longmapsto (y, x)$$

is a bialgebra pairing!

Lecture #24

$\boxed{\text{Lemma 5: The pairing } \bar{U}_q^{\leq} \times U_q^{\geq} \to \Bbbk \text{ descends to } U_q^{\leq} \times U_q^{\geq} \to \Bbbk.}$

▶ Suffices to show $(y_1 \cdot u_{ij} \cdot y_2, x) = 0 \quad \forall y_1, y_2 \in \bar{U}_q^{\leq}, \ \forall x \in U_q^{\geq}, \ \forall i \neq j$, since $\mathrm{Ker}(\bar{U}_q^{\leq} \to U_q^{\leq})$ is the 2-sided ideal generated by $\{u_{ij} \mid i \neq j\}$.

However, due to Lemma 4, it suffices to prove $(u_{ij}, x) = 0 \quad \forall x$.

Suffices to treat homogeneous x. If $\mathrm{wt}(x) \neq -\mathrm{wt}(u_{ij}) = (1-a_{ij})\alpha_i + \alpha_j$, then the equality is obvious. Otherwise, it suffices to prove $(u_{ij}, E_p x') = 0 \ \forall x'$ s.t. $\alpha_p + \mathrm{wt}(x') = (1-a_{ij})\alpha_i + \alpha_j$. But: $(u_{ij}, E_p x') = (\Delta(u_{ij}), x' \otimes E_p)$.

Recall: $\Delta(u_{ij}) = u_{ij} \otimes K_i^{-1+a_{ij}} K_j^{-1} + 1 \otimes u_{ij}$ by [Hwk 3, Problem 4]

$(u_{ij}, E_p) = 0 = (u_{ij}, x')$ if $\mathrm{wt}(x') \neq (1-a_{ij})\alpha_i + \alpha_j$

$\Rightarrow (u_{ij}, x) = 0 \quad \forall x \in U_q^{\geq}$. ∎

Thus, we conclude that there is a bilinear pairing

$$\boxed{\begin{gathered}(\ ,\ ): U_q^{\leq} \times U_q^{\geq} \to \Bbbk \quad \text{s.t.} \\ (y, xx') = (\Delta(y), x' \otimes x), \qquad (yy', x) = (y \otimes y', \Delta(x)) \\ (K_\mu, K_\nu) = q^{-(\mu,\nu)}, \quad (F_i, E_j) = \delta_{ij} \cdot \frac{1}{q_i^{-1} - q_i}, \quad (K_\mu, E_i) = 0 = (F_i, K_\mu)\end{gathered}}$$

$$\boxed{\text{Lemma 6: } (F_i^n, E_i^n) = (-1)^n q_i^{\frac{n(n-1)}{2}} \cdot \frac{[n]_{q_i}!}{(q_i - q_i^{-1})^n} \quad \forall i \in I, n \in \mathbb{N}}$$

▶ Proof is by induction on $n \geq 1$. Base ($n=1$ case): obvious.

Induction step ($n < k \rightsquigarrow n = k$):

$$(F_i^k, E_i^k) = (F_i^{k-1} \otimes F_i, \Delta(E_i^k)) \xlongequal{sl_2\text{-computation}} \Big(F_i^{k-1} \otimes F_i, \sum_{r=0}^{k} q_i^{r(k-r)} \begin{bmatrix} k \\ r \end{bmatrix}_{q_i} E_i^{k-r} K_i^r \otimes E_i^r\Big)$$

$$\xlongequal{\text{degree reasons}} (F_i^{k-1} \otimes F_i, q_i^{k-1} \cdot [k]_{q_i} \cdot E_i^{k-1} K_i \otimes E_i) = (F_i^{k-1}, E_i^{k-1}) \cdot (F_i, E_i) \cdot q_i^{k-1} \cdot [k]_{q_i}$$

$$\xlongequal{\text{Induction hypothesis}} (-1)^{k-1} \cdot q_i^{\frac{(k-1)(k-2)}{2}} \cdot \frac{[k-1]_{q_i}!}{(q_i - q_i^{-1})^{k-1}} \cdot q_i^{k-1} \cdot [k]_{q_i} \cdot \frac{-1}{q_i - q_i^{-1}} =$$

$$= (-1)^k \cdot q_i^{\frac{k(k-1)}{2}} \cdot [k]_{q_i}! / (q_i - q_i^{-1})^k$$ ∎