Lecture #25

Last time we constructed a pairing $U_q^{\le} \times U_q^{\ge} \longrightarrow \Bbbk$ s.t.

$$(y, x_1 x_2) = (\Delta(y), x_2 \otimes x_1) \quad \text{and} \quad (y_1 y_2, x) = (y_1 \otimes y_2, \Delta(x))$$

which becomes a bialgebra pairing in the sense of Lecture 15 after the swap of factors.

Similarly to our proof of [Lecture 24, Lemma 6], the above f-las allow one to compute (y,x) iteratively by using Q-grading reasoning and f-las for coproduct. To this end recall [Lecture 18, Lemma 1]:

$$\boxed{\Delta((U_q^+)_\mu) \subseteq \bigoplus_{0 \le \nu \le \mu} (U_q^+)_{\mu-\nu} K_\nu \otimes (U_q^+)_\nu}$$

As $(U_q^+)_{\alpha_i} = \Bbbk \cdot E_i$ for any simple root α_i, one can introduce linear maps

$$\boxed{r_i,\ r_i' : U_q^+ \longrightarrow U_q^+ \qquad \forall i \in I}$$

via the following equality:

$$\boxed{\Delta(x) = x \otimes 1 + \sum_{i \in I} r_i(x) K_i \otimes E_i + \ldots + \sum_{i \in I} E_i K_{\mu-\alpha_i} \otimes r_i'(x) + K_\mu \otimes x \qquad \forall x \in (U_q^+)_\mu}$$

The reasoning behind this definition is precisely part (b) of the next lemma:

Lemma 1: a) For $x \in (U_q^+)_\mu$, $x' \in (U_q^+)_{\mu'}$, we have

$$r_i(xx') = x \cdot r_i(x') + q^{(\alpha_i, \mu')} r_i(x) \cdot x'$$
$$r_i'(xx') = q^{(\alpha_i, \mu)} x \cdot r_i'(x') + r_i'(x) \cdot x'.$$

b) $(F_i y, x) = (F_i, E_i) \cdot (y, r_i'(x))$, $(y F_i, x) = (F_i, E_i) \cdot (y, r_i(x))$

c) $r_i' = \sigma r_i \sigma$ with $\sigma : U_q \circlearrowleft$ being the antiautomorphism mapping $E_i \mapsto E_i$, $F_i \mapsto F_i$, $K_i^{\pm 1} \mapsto K_i^{\mp 1}$

▷ a) If $\Delta(x) = x \otimes 1 + \sum_{i \in I} r_i(x) K_i \otimes E_i + \ldots$, $\Delta(x') = x' \otimes 1 + \sum_{i \in I} r_i(x') K_i \otimes E_i + \ldots$,

then $\Delta(xx') = \Delta(x)\Delta(x') = xx' \otimes 1 + \sum_{i \in I} \underbrace{(x r_i(x') K_i + r_i(x) K_i x')}_{(x r_i(x') + q^{(\alpha_i, \mu')} r_i(x) x') K_i} \otimes E_i + \ldots$

which proves the 1st f-la in a). The 2nd f-la is established similarly.

b) $(F_i y, x) = (F_i \otimes y, \ldots + \sum_j E_j K_{\mu - \alpha_j} \otimes r_j'(x) + \underset{\mu = wt(x)}{K_\mu} \otimes x) \overset{\text{degree}}{\underset{\text{reasons}}{=}} \underbrace{(F_i, E_i K_{\mu-\alpha_i})}_{=(F_i, E_i)} \cdot (y, r_i'(x))$

which proves the 1st f-la in (b). The 2nd f-la is proved similarly.

[Exercise: Prove part c)

The above construction can be likewise applied to U_q^-. To this end, we define

$$\boxed{r_i, r_i' : U_q^- \longrightarrow U_q^- \quad \forall i \in I}$$

via the following equality:

$$\boxed{\Delta(y) = y \otimes K_{-\mu} + \sum_{i\in I} r_i(y) \otimes F_i K_{-(\mu-\alpha_i)} + \dots + \sum_{i \in I} F_i \otimes r_i'(y) K_i^{-1} + 1 \otimes y \qquad \forall y \in (U_q^-)_{-\mu}}$$

The following is the analogue of Lemma 1:

<u>Lemma 2</u>: a) For $y \in (U_q^-)_{-\mu}$, $y' \in (U_q^-)_{-\mu'}$, we have

$$r_i(yy') = q^{(\alpha_i, \mu')} y r_i(y') + r_i(y) \cdot y'$$
$$r_i'(yy') = y r_i'(y') + q^{(\alpha_i, \mu')} r_i'(y) y'$$

b) $(y, E_i x) = (F_i, E_i) \cdot (r_i(y), x)$, $\quad (y, x E_i) = (F_i, E_i) \cdot (r_i'(y), x)$

c) $r_i' = \sigma r_i \sigma$

d) We have $r_i(y) = \omega r_i' \omega(y)$ and $r_i'(y) = \omega r_i \omega(y)$ $\forall y \in U_q^-$,

where $\omega : U_q \longrightarrow U_q$ is the Cartan involution mapping $E_i \mapsto F_i$, $F_i \mapsto E_i$, $K_i^{\pm 1} \mapsto K_i^{\mp 1}$

▸ a)-c) are proved entirely analogously to Lemma 1.

The equalities of d) hold for $y = F_j$ $\forall j \in I$, hence, it suffices to check that if d) holds for y_1 & y_2, then it also holds for $y_1 \cdot y_2$. This is straightforward:

$$\omega r_i' \omega(y_1 y_2) = \omega r_i'(\omega(y_1) \cdot \omega(y_2)) \overset{\text{Lemma 1}}{=\!=\!=} \omega(q^{(\alpha_i, \mu)} \omega(y_1) \cdot r_i' \omega(y_2) + r_i' \omega(y_1) \cdot \omega(y_2))$$

$$= q^{(\alpha_i, \mu)} \cdot \underbrace{\omega^2(y_1)}_{=y_1} \cdot \underbrace{\omega r_i' \omega(y_2)}_{=r_i(y_2)} + \underbrace{\omega r_i' \omega(y_1)}_{=r_i(y_1)} \cdot \underbrace{\omega^2(y_2)}_{=y_2} \overset{\text{Lemma 2a)}}{=\!=\!=} r_i(y_1 y_2)$$

The 2^{nd} f-la in d) is checked similarly ∎

<u>Lemma 3</u>: $(y, x) = (\omega(x), \omega(y))$ $\forall y \in U_q^-$, $x \in U_q^+$

▸ Can assume $x \in (U_q^+)_\mu$, $y \in (U_q^-)_{-\mu}$. Moreover, suffices to treat $y = F_i y'$, $y' \in (U_q^-)_{-\mu+\alpha_i}$:

$$(F_i y', x) \overset{\text{Lemma 1}}{=\!=\!=} (F_i, E_i) \cdot (y', r_i'(x)) \overset{\substack{\text{Assumption}\\ \text{of induction}}}{=\!=\!=} (F_i, E_i) \cdot (\omega r_i'(x), \omega(y')) \overset{\text{Lemma 2d)}}{=\!=\!=}$$

$$(F_i, E_i) \cdot (r_i \omega(x), \omega(y')) \overset{\text{Lemma 2}}{=\!=\!=} (\omega(x), \underbrace{E_i}_{=\omega(F_i)} \omega(y')) = (\omega(x), \omega(y))$$

Thus, the proof proceeds by induction on the height of $\mu \in Q$ ∎

[**Exercise**: Prove that $(S(y), S(x)) = (y, x)$ $\forall y \in U_q^{\leq}, x \in U_q^{\geq}$, where S — antipode.

More importantly, the maps r_i, r_i' also arise when commuting elements of U_q^- with E_i, or el-s of U_q^+ with F_i, as shown in the next result:

Lemma 4: For $i \in I$, $y \in (U_q^-)_{-\mu}$, $x \in (U_q^+)_\mu$, we have:

$$E_i y - y E_i = \frac{1}{q_i - q_i^{-1}} \left(K_i r_i(y) - r_i'(y) K_i^{-1} \right)$$

$$F_i x - x F_i = \frac{-1}{q_i - q_i^{-1}} \left(r_i(x) K_i - K_i^{-1} r_i'(x) \right)$$

▶ First, we prove the first equality. It's obvious for $y = F_j$. It thus remains to check that it holds for $y_1 \cdot y_2$ if it holds for y_1 & y_2:

$$[E_i, y_1 \cdot y_2] = [E_i, y_1] \cdot y_2 + y_1 \cdot [E_i, y_2] = \frac{1}{q_i - q_i^{-1}} \left(K_i r_i(y_1) y_2 - r_i'(y_1) K_i^{-1} y_2 \right) +$$

$$+ \frac{1}{q_i - q_i^{-1}} \left(y_1 K_i r_i(y_2) - y_1 r_i'(y_2) K_i^{-1} \right) = \frac{1}{q_i - q_i^{-1}} K_i \Big(\underbrace{r_i(y_1) y_2 + q^{(\alpha_i, wt\, y_1)} y_1 \cdot r_i(y_2)}_{= r_i(y_1 y_2)} \Big) -$$

$$- \frac{1}{q_i - q_i^{-1}} \Big(\underbrace{r_i'(y_1) \cdot y_2 \cdot q^{(\alpha_i, wt\, y_2)} + y_1 r_i'(y_2)}_{= r_i'(y_1 y_2)} \Big) K_i^{-1}$$

To deduce the second equality in the lemma, apply Cartan involution ω to the 1st one with $y = \omega(x)$, and use Lemma 2d). ■

The above result implies also:

Lemma 5: For any $i \in I$, $\lambda \in Q$, $x \in (U_q^+)_\mu$, $y \in (U_q^-)_{-\nu}$, we have:

$$ad(E_i)(y K_\lambda x) = y K_\lambda \left(q^{-(\lambda, \alpha_i)} E_i x - q^{(\mu - \nu, \alpha_i)} x E_i \right) + \frac{\left(q^{-(\nu - \alpha_i, \alpha_i)} r_i(y) K_{\lambda + \alpha_i} - r_i'(y) K_{\lambda - \alpha_i} \right) x}{q_i - q_i^{-1}}$$

$$ad(F_i)(y K_\lambda x) = q^{-(\mu, \alpha_i)} \left(F_i y - q^{-(\lambda, \alpha_i)} y F_i \right) K_{\lambda + \alpha_i} x + \frac{y \left(q^{-(\mu - \alpha_i, \alpha_i)} K_\lambda r_i'(x) - q^{-2(\mu - \alpha_i, \alpha_i)} K_{\lambda + 2\alpha_i} r_i(x) \right)}{q_i - q_i^{-1}}$$

[**Exercise**: Prove this lemma.

Lecture #25

We conclude with the following main result for today:

> **Proposition 1**: Assuming "tQ-condition" (see Lecture 23), the restriction $(\cdot,\cdot): (U_q^-)_{-\mu} \times (U_q^+)_\mu \to \Bbbk$ is non-degenerate $\forall \mu \geq 0$.

▶ As $\dim (U_q^-)_{-\mu} = \dim (U_q^+)_\mu$, due to Cartan involution, it suffices to show $(y, (U_q^+)_\mu) = 0 \Rightarrow y = 0$. The proof is by induction on the height of μ. Applying Lemma 2 to $(y, E_i \cdot x')$ and $(y, x' \cdot E_i)$ for any $i \in I$ and $x' \in (U_q^+)_{\mu - \alpha_i}$ and using the assumption of induction, we find $r_i(y) = 0 = r_i'(y)$ $\forall i$.

But then, $[y, E_i] = 0$ $\forall i \in I$, due to Lemma 4. Applying ω to this equality, we see that $x := \omega(y) \in (U_q^+)_\mu$ commutes with all F_i.

> **Claim**: If $x \in (U_q^+)_\mu$ $(\mu > 0)$ commutes with all $\{F_i\}_{i \in I}$, then $x = 0$

▷ For any $\lambda \in P$, consider the Verma module $M(\lambda)$ with the highest weight vector v_λ. For degree reasons: $x(v_\lambda) = 0$. But as $M(\lambda)$ is generated from v_λ by $\{F_j\}$ which commute with x, we conclude that $x|_{M(\lambda)} = 0$. Hence, x acts trivially on all simple finite-dimensional modules.

But by [Lecture 22, Theorem 2], under "tQ-condition" any finite-dimen. $U_q(\mathfrak{g})$-module is $\simeq \oplus$ simple f.d, hence, x acts trivially on all f.d. mod. The latter implies the stated $x = 0$, due to [Hwk 4, Problem 4],

cf. our proof of [Lecture 23, Lemma 2b)]. □

This establishes the above Claim, thus completing the proof of Prop. ■

Combining this result and Lemma 3, we obtain:

> **Corollary 1**: The bilinear form
> $$(\cdot,\cdot)': U_q^+ \times U_q^+ \to \Bbbk \text{ given by } (x_1, x_2)' = (\omega(x_1), x_2)$$
> is symmetric and non-degenerate.