Lecture #27

Goal: Prove the quantum analogue of Harish-Chandra isomorphism (see the very last result in Lecture #23).

Recall that back in Lecture #23, we constructed an embedding

$$\boxed{\gamma_{-\rho}\circ\pi =: HC: \ Z_q(\mathfrak{g}) \hookrightarrow (U^0_{ev})^W}$$

and the claim is that it's actually an isomorphism. To prove this, we shall construct a family of central elements using the pairing $\langle\,,\,\rangle$.

- The following intermediate result replicates that for fin. dim. vector space with a non-degenerate pairing, an elt is the same as a functional

Lemma 1 (Assuming "+Q-conditions"): Let M be a fin. dim. type 1 $U_q(\mathfrak{g})$-mod such that all weights λ of M satisfy $2\lambda\in Q$. Then for any $m\in M, f\in M^*$ $\exists!\ u\in U_q(\mathfrak{g})$ s.t. the matrix coefficient $c_{f,m}$ equals $\langle -,u\rangle$. Here, $c_{f,m}(v) := f(v(m))$

▸ By linearity, we can assume that m and f are homogeneous, that is: $m\in M_\lambda$, $f\in(M^*)_{-\lambda'}$ for some $\lambda,\lambda'\in P$, where the latter means $f(M_{\lambda''\neq\lambda'})=0$

First, we note that $c_{f,m}|_{(U_q^-)_{-\mu}\cdot U_q^0\cdot(U_q^+)_\nu} = 0$ unless $\lambda'=\lambda+\nu-\mu$ (for degree reasons)

Second, since M is fin. dim the set $\{\nu\in Q_+ \mid (U_q^+)_\nu m\neq 0\}$ is finite.

Thus, evoking the triangular decomposition $U_q(\mathfrak{g}) = \bigoplus_{\mu,\nu\in Q_+}(U_q^-)_{-\mu}U_q^0(U_q^+)_\nu$ we conclude that the matrix coefficient $c_{f,m}$ is nonzero only on finitely many of these summands.

Fix μ,ν as above. Then $\forall\, x\in(U_q^+)_\nu,\ y\in(U_q^-)_{-\mu},\ \eta\in Q$, we have:

$$c_{f,m}(yK_\mu\cdot K_\eta\cdot x) = f(yK_\mu K_\eta x(m)) = q^{(\eta,\lambda+\nu)}f(yK_\mu xm) = f(yK_\mu xm)\cdot(q^{1/2})^{(\eta,2\lambda+2\nu)}$$

But: the assignment $(U_q^-)_{-\mu}\times(U_q^+)_\nu\to\Bbbk$, $(y,x)\mapsto f(yK_\mu xm)$, is bilinear

$2\lambda\in Q \Rightarrow 2\lambda+2\nu\in Q$

$\Rightarrow$ by [Lecture #26, Lemma 1] $c_{f,m}(v) = \langle v, u_{\mu\nu}\rangle$ $\forall v\in(U_q^-)_{-\mu}U_q^0(U_q^+)_\nu$. Now take $u=\sum u_{\mu\nu}$ (finite sum)

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Note that the uniqueness in Lemma 1 is due to $\langle , \rangle$ being nondegenerate.
With the above lemma in hand, we can now produce many central el-s.

Proposition 1 ("tQ-condition"): Let λ be a dominant integral weight s.t. $2\lambda \in Q$

(a) $\exists !\, z_\lambda \in U_q(\mathfrak{g})$ s.t. $\langle u, z_\lambda \rangle = Tr_{L(\lambda)}(u K_{-2\rho}) \quad \forall u \in U_q(\mathfrak{g})$

(b) z_λ is central

▶ (a) Pick any basis $\{w_i\}$ of $L(\lambda)$ and let $\{w_i^*\}$ be the dual basis of $L(\lambda)^*$.
Then: $Tr_{L(\lambda)}(u K_{-2\rho}) = \sum_i c_{w_i^*, K_{-2\rho}(w_i)}(u)$ – sum of matrix coeff-s, hence, by Lemma 1 it can be written as $\langle u, z_\lambda \rangle$ for some z_λ. Uniqueness is again due to $\langle , \rangle$ - nondegenerate

(b) To check $z_\lambda \in Z_q(\mathfrak{g})$, we start with the following

Claim: The linear map ("quantum trace") $U_q(\mathfrak{g}) \longrightarrow \mathbb{k},\ u \mapsto tr_{L(\lambda)}(u K_{-2\rho})$, is a $U_q(\mathfrak{g})$-module morphism $\left(\begin{array}{l}\text{adjoint action on the left,}\\ \text{counit action on the right}\end{array}\right)$

[Exercise: Prove this claim

Let's now finish the proof of $z_\lambda \in Z_q(\mathfrak{g})$. Indeed, we get:

$$\begin{array}{ccc} \varepsilon(u)\cdot\langle v, z_\lambda\rangle & = & \langle v, \varepsilon(u) z_\lambda\rangle \\ \| & & \\ \langle \mathrm{ad}(u)v, z_\lambda\rangle & = & \langle v, \mathrm{ad}(S(u)) z_\lambda\rangle \end{array} \quad \forall u, v \in U_q(\mathfrak{g}) \underset{\langle , \rangle\text{-nondeg}}{\Longrightarrow} \boxed{\varepsilon(u) z_\lambda = \underset{\forall u}{\mathrm{ad}(S(u)) z_\lambda}}$$

Since $\varepsilon(S(u)) = \varepsilon(u)$, we get $\boxed{\mathrm{ad}(u) z_\lambda = \varepsilon(u) z_\lambda \ \forall u \in U_q(\mathfrak{g})}$

- take $u = K_i \Rightarrow K_i z_\lambda K_i^{-1} = z_\lambda \Rightarrow z_\lambda$ commutes with K_i
- take $u = F_i \Rightarrow 0 = \mathrm{ad}(F_i) z_\lambda = [F_i, z_\lambda] K_i \Rightarrow z_\lambda$ commutes with F_i
- take $u = E_i \Rightarrow 0 = \mathrm{ad}(E_i) z_\lambda = E_i z_\lambda - K_i z_\lambda K_i^{-1} E_i = [E_i, z_\lambda] \Rightarrow z_\lambda$ commutes with E_i

$\Rightarrow z_\lambda \in Z_q(\mathfrak{g})$ ∎

As z_λ commutes with all K_i and $q \neq \sqrt[?]{1}$, we can write

$$z_\lambda = \sum_{\mu \in Q_+} z_{\lambda,\mu} \quad \text{with } z_{\lambda,\mu} \in (U_q^-)_{-\mu} \cdot U_q^0 \cdot (U_q^+)_\mu$$

Consider $z_{\lambda,0} = \pi(z_\lambda)$ and decompose it w.r.t. basis $\{K_\nu\}$ of U_q^0:

$$z_{\lambda,0} = \sum_\nu a_\nu K_\nu$$

Then: $\langle K_\mu, z_\lambda \rangle = \langle K_\mu, z_{\lambda,0} \rangle = \sum_\nu a_\nu \langle K_\mu, K_\nu \rangle = \sum_\nu a_\nu (q^{-1/2})^{(\mu,\nu)}$

$\| \text{Prop 1}$

$$tr_{L(\lambda)}(K_{\mu-2\rho}) = \sum_{\lambda'} \dim L(\lambda)_{\lambda'} \cdot q^{(\lambda', \mu-2\rho)} = \sum_{\lambda'} \dim L(\lambda)_{\lambda'} \cdot q^{-2(\lambda',\rho)} (q^{-1/2})^{(-2\lambda',\mu)}$$

As above holds $\forall \mu$, we again get (using Artin's lemma):

$$a_\nu = \dim L(\lambda)_{-\frac{\nu}{2}} \cdot q^{(\nu,\rho)}$$

$$\Downarrow$$

$$\pi(z_\lambda) = z_{\lambda,0} = \sum_\nu \dim L(\lambda)_{-\nu/2} \cdot q^{(\nu,\rho)} \cdot K_\mu$$

$$\Downarrow$$

$$\boxed{HC(z_\lambda) = \gamma_{-\rho} \circ \pi(z_\lambda) = \sum_\nu \dim L(\lambda)_{-\frac{\nu}{2}} K_\nu}$$

Now we are ready do prove the main result:

$$\boxed{\text{Theorem 1 (Assuming "}\tfrac{1}{2}Q\text{-condition"): } HC: Z_q(\mathfrak{g}) \xrightarrow{\sim} (U_{ev}^0)^W \text{ - isom}}$$

- Know (from Lecture 23): above is an embedding
- Also it's clear that $(U_{ev}^0)^W$ is spanned by $Av(\mu) := \sum_{\nu \in W(\mu)} K_\nu$ as μ ranges through $\frac{1}{2}Q_+ \cap P_+$.
- Finally, $\forall \mu \in Q_+ \cap 2P_+$ set $\lambda := \mu/2$. Then $z_\lambda \in Z_q(\mathfrak{g})$ and the above computation shows that

$$HC(z_\lambda) = Av(-\mu) + \sum_{\substack{\nu < \lambda \\ \nu\text{-dominant}}} \underline{\text{coeff}} \cdot Av(-2\nu)$$

Hence, by induction on the height of μ, we get $Av(-\mu) \in \text{Im}(HC)$.
Thus, HC is also surjective.