Lecture #28

Goal for today: Construct a functorial isomorphism $M_2 \otimes M_1 \xrightarrow{\sim} M_1 \otimes M_2$ of $U_q(\mathfrak{g})$-modules for any two type 1 finite dimensional modules M_1, M_2, generalizing $\mathfrak{sl}_2 = \mathfrak{g}$ case from Lectures 13-14.

• Recall that for $\mathfrak{g} = \mathfrak{sl}_2$, our $\Theta = \sum_{n \geq 0} \Theta_n$ with $\Theta_n = a_n \cdot F^n \otimes E^n$. Here, F^n and E^n span dual weight subspaces $(U_q^-(\mathfrak{sl}_2))_{-n\alpha}$ and $(U_q^+(\mathfrak{sl}_2))_{n\alpha}$, α-simple root. Moreover, evoking $a_n = (-1)^n \cdot \frac{(q-q^{-1})^n}{[n]_q!} \cdot q^{-\frac{n(n-1)}{2}}$ as well as [Lecture 24, Lemma 6], we get $a_n = \frac{1}{(F^n, E^n)}$, where $(\ ,\ ) : U_q^{\leq}(\mathfrak{sl}_2)_{-n\alpha} \times U_q^{\geq}(\mathfrak{sl}_2)_{n\alpha} \to \mathbb{k}$ is the pairing of Lecture 24.

This observation indicates the generalization for a general simple $\mathfrak{g}$. For every $\mu \in Q_+$ (i.e. $\mu = \sum_{i \in I} n_i \alpha_i$ with $n_i \in \mathbb{Z}_{\geq 0}$) choose an arbitrary basis $\{x_i^\mu\}$ ($1 \leq i \leq N_\mu := \dim (U_q^+)_\mu$) of $(U_q^+)_\mu$ and let $\{y_i^\mu\}$ be the dual basis of $(U_q^-)_{-\mu}$ w.r.t. the non-degenerate pairing $(\cdot, \cdot) : (U_q^{\leq})_{-\mu} \times (U_q^{\geq})_\mu \to \mathbb{k}$ of Lecture 24:

$$\boxed{(y_i^\nu, x_j^\mu) = \delta_{\nu\mu} \cdot \delta_{ij}}$$

We then define

$$\boxed{\Theta_\mu := \sum_{i=1}^{N_\mu} y_i^\mu \otimes x_i^\mu \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})}$$

As always, this construction is independent of a basis.

Lemma 1: (a) Let σ be the antiautomorphism of $U_q(\mathfrak{g})$ s.t. $\sigma: E_i \mapsto F_i,\ F_i \mapsto E_i,\ K_i^{\pm 1} \mapsto K_i^{\mp 1}$. Then: $(\sigma \otimes \sigma)(\Theta_\mu) = \Theta_\mu$

(b) Let ω be the Cartan involution of $U_q(\mathfrak{g})$. Then:

$$(\omega \otimes \omega)(\Theta_\mu) = \underset{\text{flip map}}{\tau}(\Theta_\mu) =: \Theta_\mu^{op}$$

▷ (a) Clearly $\{\sigma(x_i^\mu)\}_{i=1}^{N_\mu}$ – another basis of $(U_q^+)_\mu$, $\{\sigma(y_i^\mu)\}_{i=1}^{N_\mu}$ – another basis of $(U_q^-)_{-\mu}$. It remains to use $(\sigma(y), \sigma(x)) = (y, x)$ that can be proved by induction on μ.

(b) Clearly $\{\omega(x_i^\mu)\}_{i=1}^{N_\mu}$ – basis of $(U_q^-)_{-\mu}$, $\{\omega(y_i^\mu)\}_{i=1}^{N_\mu}$ – basis of $(U_q^+)_\mu$. It remains to use $(\omega(x), \omega(y)) = (y, x)$ from [Lecture 25, Lemma 3] ▯

Lecture #28

For $\mu \in Q \setminus Q_+$, set $\Theta_\mu := 0$. We thus get a family $\{\Theta_\mu\}_{\mu \in Q}$.

Lemma 2: For any $\mu \in Q_+$, $i \in I$, we have:

a) $(K_j \otimes K_j)\Theta_\mu = \Theta_\mu (K_j \otimes K_j)$

b) $(E_j \otimes 1)\Theta_\mu + (K_j \otimes E_j)\Theta_{\mu-\alpha_j} = \Theta_\mu (E_j \otimes 1) + \Theta_{\mu-\alpha_j}(K_j^{-1} \otimes E_j)$

c) $(1 \otimes F_j)\Theta_\mu + (F_j \otimes K_j^{-1})\Theta_{\mu-\alpha_j} = \Theta_\mu (1 \otimes F_j) + \Theta_{\mu-\alpha_j}(F_j \otimes K_j)$

← generalization of Lemma 3 from Lecture 13

▶ a) Obvious (as $\deg y^\mu_i + \deg x^\mu_i = 0$)

b) Let's compute $[(E_j \otimes 1), \Theta_\mu]$:

$$(E_j \otimes 1)\Theta_\mu - \Theta_\mu (E_j \otimes 1) = \sum_i (E_j y^\mu_i - y^\mu_i E_j) \otimes x^\mu_i \overset{\text{Lecture 25}}{\underset{\text{Lemma 4}}{=}} \sum_i \frac{1}{q_j - q_j^{-1}} \left(K_j r_j(y^\mu_i) - r_j'(y^\mu_i) K_j^{-1}\right) \otimes x^\mu_i \quad (1)$$

But: we have $x = \sum_k (y^\nu_k, x) \cdot x^\nu_k$, $y = \sum_k (y, x^\nu_k) \cdot y^\nu_k$ $\forall x \in (U_q^+)_\nu, y \in (U_q^-)_{-\nu}$.

$$\overset{(1)}{=} \frac{1}{q_j - q_j^{-1}} \sum_i \left(K_j \sum_{k=1}^{N_{\mu-\alpha_j}} (r_j(y^\mu_i), x^{\mu-\alpha_j}_k) y^{\mu-\alpha_j}_k - \sum_k (r_j'(y^\mu_i), x^{\mu-\alpha_j}_k) y^{\mu-\alpha_j}_k \cdot K_j^{-1} \right) \otimes x^\mu_i =$$

$$\overset{\text{Lecture 25}}{\underset{\text{Lemma 2}}{=}} \sum_{i=1}^{N_\mu} \left(-K_j \sum_k (y^\mu_i, E_j x^{\mu-\alpha_j}_k) y^{\mu-\alpha_j}_k + \sum_k (y^\mu_i, x^{\mu-\alpha_j}_k \cdot E_j) y^{\mu-\alpha_j}_k K_j^{-1} \right) \otimes x^\mu_i \;=$$

But: Again use the above decomposition of a vector w.r.t. dual bases

$$= \sum_k \left(y^{\mu-\alpha_j}_k K_j^{-1} \otimes x^{\mu-\alpha_j}_k E_j - K_j y^{\mu-\alpha_j}_k \otimes E_j x^{\mu-\alpha_j}_k \right) = \Theta_{\mu-\alpha_j}(K_j^{-1} \otimes E_j) - (K_j \otimes E_j)\Theta_{\mu-\alpha_j}$$

c) Analogous computation

[Exercise: Work out details

■

Let now M_1, M_2 be two type 1 finite dimensional $U_q(\mathfrak{g})$-modules.
Then $M_1 \otimes M_2 = \sum_{\lambda_1, \lambda_2 \in P} (M_1)_{\lambda_1} \otimes (M_2)_{\lambda_2}$ and all but finitely many terms are zero.
For any $\mu \in Q_+$, we get $\Theta_\mu : M_1 \otimes M_2 \circlearrowleft$ s.t. $\Theta_\mu((M_1)_{\lambda_1} \otimes (M_2)_{\lambda_2}) \subseteq (M_1)_{\lambda_1 - \mu} \otimes (M_2)_{\lambda_2 + \mu}$
By above (as both M_1, M_2 — fin. dim.), we conclude that

$$\boxed{\Theta = \Theta_{M_1, M_2} = \sum_{\mu \in Q_+} \Theta_\mu : M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2}$$

is well-defined. Moreover, it's clearly an invertible operator (since choosing an appropriate basis, it's represented by unipotent matrix)

Then, we can rephrase Lemma 2 as follows:

$$\boxed{\Delta(u)\circ \Theta = \Theta \circ \tilde{\Delta}(u)} \quad \longleftarrow \text{cf. Lecture 14 for } \mathfrak{g}=\mathfrak{sl}_2$$

where $\tilde{\Delta}$ is the "twisted comultiplication" on $U_q(\mathfrak{g})$ determined by

$$\boxed{\tilde{\Delta}: K_j \mapsto K_j\otimes K_j\,,\quad E_j\mapsto E_j\otimes 1 + K_j^{-1}\otimes E_j\,,\quad F_j\mapsto 1\otimes F_j + F_j\otimes K_j \quad \forall j\in I}$$

Note: $\tilde{\Delta} = (\sigma\otimes\sigma)\circ\Delta\circ\sigma$

- We shall also pick

$$\boxed{f: P\times P \longrightarrow \mathbb{k}^* \quad \text{s.t.} \begin{cases} f(\lambda+\nu,\mu) = q^{-(\nu,\mu)} f(\lambda,\mu) \\ f(\lambda,\mu+\nu) = q^{-(\nu,\lambda)} f(\lambda,\mu)\end{cases} \quad \forall \lambda,\mu\in P \quad \forall \nu\in Q}$$

This allows to define

$$\boxed{\begin{aligned}\tilde{f}: M_1\otimes M_2 &\longrightarrow M_1\otimes M_2\\ m_1\otimes m_2 &\longmapsto f(\lambda,\mu)\cdot m_1\otimes m_2 \qquad \forall \lambda,\mu \quad \forall m_1\in (M_1)_\lambda,\ m_2\in(M_2)_\mu\end{aligned}}$$

[Exercise: Classify all f as above.

The following result is proved exactly as its $\mathfrak{sl}_2$-version ([Lecture 14, Thm 1]).

Theorem 1: For any finite-dimensional $U_q(\mathfrak{g})$-modules M_1, M_2 of type 1

$$\Theta\circ\tilde{f}\circ\tau = \Theta^f\circ\tau : M_2\otimes M_1 \longrightarrow M_1\otimes M_2$$

is an isomorphism of $U_q(\mathfrak{g})$-modules

[Exercise: Work out details by verifying $\tilde{\Delta}(u)\circ\tilde{f} = \tilde{f}\circ\Delta^{op}(u)$ $\forall u\in U_q(\mathfrak{g})$.

Let's now have three type 1 $U_q(\mathfrak{g})$-modules M_1, M_2, M_3. Fix f as above, and consider the corresponding $\Theta^f_{12}, \Theta^f_{13}, \Theta^f_{23}: M_1\otimes M_2\otimes M_3 \longrightarrow M_1\otimes M_2\otimes M_3$.

The following result generalizes its $\mathfrak{sl}_2$-version from [Lecture 14, Thm 2]:

Theorem 2: $\Theta^f_{12}\circ\Theta^f_{13}\circ\Theta^f_{23} = \Theta^f_{23}\circ\Theta^f_{13}\circ\Theta^f_{12}$

While similar, the proof is a bit more technical and we shall highlight key spots

Lecture #28

Lemma 3: For any $\mu \in Q_+$, we have:

a) $(\Delta \otimes \mathrm{id})\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_{\mu-\nu})_{23}\,(1 \otimes K_\nu^{-1} \otimes 1)\,(\Theta_\nu)_{13}$

b) $(\mathrm{id} \otimes \Delta)\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_{\mu-\nu})_{12}\,(1 \otimes K_\nu \otimes 1)\,(\Theta_\nu)_{13}$

$\leftarrow$ For $\mathfrak{g} = \mathfrak{sl}_2$, this was established in the very end of Lecture #14. (proof of Lemma 3)

The proof is based on the following simple observation:

Claim: $\Delta(x) = \sum_{0 \le \nu \le \mu} \sum_{i,j} (y_i^{\mu-\nu} y_j^\nu, x)\, x_i^{\mu-\nu} K_\nu \otimes x_j^\nu \qquad \forall x \in (U_q^+)_\mu$

$\Delta(y) = \sum_{0 \le \nu \le \mu} \sum_{i,j} (y, x_i^{\mu-\nu} x_j^\nu)\, y_j^\nu \otimes y_i^{\mu-\nu} K_\nu^{-1} \qquad \forall y \in (U_q^-)_{-\mu}$

$\triangleright$ To prove the first equality, we note that $\Delta(x) = \sum_{0 \le \nu \le \mu} \sum_{i,j} c_{ij}^\nu\, x_i^{\mu-\nu} K_\nu \otimes x_j^\nu$ with the constants c_{ij}^ν determined via

$c_{ij}^\nu = (y_i^{\mu-\nu} \otimes y_j^\nu, \Delta(x)) = (y_i^{\mu-\nu} y_j^\nu, x)$ which proves the equality.

The second f-la is proved analogously $\square$

a) Let's now prove part a) using the above Claim.

$$(\Delta \otimes \mathrm{id})\Theta_\mu = \sum_k \Delta(y_k^\mu) \otimes x_k^\mu \overset{\text{Claim}}{=} \sum_{0 \le \nu \le \mu}^{k,i,j} (y_k^\mu, x_i^{\mu-\nu} x_j^\nu)\, y_j^\nu \otimes y_i^{\mu-\nu} K_\nu^{-1} \otimes x_k^\mu =$$

$$= \sum_{0 \le \nu \le \mu}^{i,j} y_j^\nu \otimes y_i^{\mu-\nu} K_\nu^{-1} \otimes \Big(\underbrace{\sum_k (y_k^\mu, x_i^{\mu-\nu} x_j^\nu) \cdot x_k^\mu}_{= x_i^{\mu-\nu} x_j^\nu} \Big)$$

$$= \sum_{0 \le \nu \le \mu}^{i,j} y_j^\nu \otimes y_i^{\mu-\nu} K_\nu^{-1} \otimes x_i^{\mu-\nu} x_j^\nu = \sum_{0 \le \nu \le \mu} (\Theta_{\mu-\nu})_{23}\,(1 \otimes K_\nu^{-1} \otimes 1)\,(\Theta_\nu)_{13}$$

Exercise: Prove analogously part b)

Lemma 4: For any $\mu \in Q_+$, we have:

a) $(\tilde{\Delta} \otimes \mathrm{id})\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_\nu)_{13}\,(1 \otimes K_\nu \otimes 1)\,(\Theta_{\mu-\nu})_{23}$

b) $(\mathrm{id} \otimes \tilde{\Delta})\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_\nu)_{13}\,(1 \otimes K_\nu^{-1} \otimes 1)\,(\Theta_{\mu-\nu})_{12}$

Lecture #28

▶ (Proof of Lemma 4)

Recall that $\tilde{\Delta} = (\sigma\otimes\sigma)\circ\Delta\circ\sigma$ and $\sigma^2 = \mathrm{id}$. Thus:

$$(\tilde{\Delta}\otimes\mathrm{id})\Theta_\mu = (\sigma\otimes\sigma\otimes\sigma)(\Delta\otimes\mathrm{id})(\sigma\otimes\sigma)(\Theta_\mu) \overset{\text{Lemma 1}}{=\!=} (\sigma\otimes\sigma\otimes\sigma)(\Delta\otimes\mathrm{id})\,\Theta_\mu \overset{\text{Lemma 3a)}}{=\!=}$$

$$= (\sigma\otimes\sigma\otimes\sigma)\Big(\sum_{0\le\nu\le\mu}(\Theta_{\mu-\nu})_{23}\,(1\otimes K_\nu^{-1}\otimes 1)(\Theta_\nu)_{13}\Big) \overset{\text{Lemma 1}}{=\!=} \sum_{0\le\nu\le\mu}(\Theta_\nu)_{13}(1\otimes K_\nu\otimes 1)(\Theta_{\mu-\nu})_{23}$$

[Exercise: Prove analogously part b) ∎

Finally, we have the following straightforward result:

> **Lemma 5**: For any $\mu\in Q_+$, we have:
> a) $\tilde{f}_{12}\circ(\Theta_\mu)_{13} = (\Theta_\mu)_{13}\circ(1\otimes K_\mu\otimes 1)\circ\tilde{f}_{12}$
> b) $\tilde{f}_{12}\circ\tilde{f}_{13}\circ(\Theta_\mu)_{23} = (\Theta_\mu)_{23}\circ\tilde{f}_{12}\circ\tilde{f}_{13}$

← generalizing Lemmas 1-2 from Lecture 14 for $\mathfrak{g}=\mathfrak{sl}_2$

With Lemmas 3-5 at hand, we can now establish Theorem 2 exactly as for $\mathfrak{g}=\mathfrak{sl}_2$ in Lecture 14:

▶ (Proof of Theorem 2)

$$\Theta^f_{12}\circ\Theta^f_{13}\circ\Theta^f_{23} = \Theta_{12}\circ\tilde{f}_{12}\circ\Theta_{13}\circ\tilde{f}_{13}\circ\Theta_{23}\circ\tilde{f}_{23} = \Theta_{12}\circ\Big(\sum_\eta(\Theta_\eta)_{13}(1\otimes K_\eta\otimes 1)\Big)\Theta_{23}\circ\tilde{f}_{12}\tilde{f}_{13}\tilde{f}_{23}$$

Likewise:

$$\Theta^f_{23}\circ\Theta^f_{13}\circ\Theta^f_{12} = \Theta_{23}\circ\Big(\sum_\eta(\Theta_\eta)_{13}(1\otimes K_\eta^{-1}\otimes 1)\Big)\circ\Theta_{12}\circ\tilde{f}_{23}\tilde{f}_{13}\tilde{f}_{12}$$

As we obviously have $\tilde{f}_{12}\tilde{f}_{13}\tilde{f}_{23} = \tilde{f}_{23}\tilde{f}_{13}\tilde{f}_{12}\in\mathrm{End}(M_1\otimes M_2\otimes M_3)$, it remains to prove:

$$\underbrace{\Theta_{12}\circ\Big(\sum_\eta(\Theta_\eta)_{13}(1\otimes K_\eta\otimes 1)\Big)\circ\Theta_{23}}_{\overset{\text{Lemma 4}}{=\!=}(\tilde{\Delta}\otimes\mathrm{id})\Theta} = \underbrace{\Theta_{23}\circ\Big(\sum_\eta(\Theta_\eta)_{13}(1\otimes K_\eta^{-1}\otimes 1)\Big)\circ\Theta_{12}}_{\overset{\text{Lemma 3}}{=\!=}(\Delta\otimes\mathrm{id})\Theta}$$

which obviously follows from $\Delta(u)\Theta = \Theta\tilde{\Delta}(u)$ established in Lemma 2 ∎