Lecture #29

Last time: • $\Theta = \sum_{\substack{\mu \geq 0 \\ \mu \in Q}} \Theta_\mu$, $\Theta_\mu = \sum_i y_i^\mu \otimes x_i^\mu$ — dual bases of $(U_q^-)_{-\mu}, (U_q^+)_\mu$

• $f: P\times P \to \mathbb{k}^*$ s.t. $f(\lambda+\nu,\mu) = f(\lambda,\mu)\cdot q^{-(\nu,\mu)}$, $f(\lambda,\mu+\nu) = f(\lambda,\mu)\, q^{-(\nu,\lambda)}$ $\forall\, \lambda,\mu\in P,\ \nu\in Q$

• Theorem 1: $\Theta\circ \tilde{f}\circ\tau = \Theta^f\circ\tau$ provides isomorphism of $U_q(\mathfrak{g})$-modules $M_2\otimes M_1 \xrightarrow{\sim} M_1\otimes M_2$ (f.d, type 1)

• Theorem 2: $\Theta^f_{12}\circ\Theta^f_{13}\circ\Theta^f_{23} = \Theta^f_{23}\circ\Theta^f_{13}\circ\Theta^f_{12}$: $M_1\otimes M_2\otimes M_3 \to M_1\otimes M_2\otimes M_3$.

The last result from sl_2-setup of Lectures 14-15 that admits a direct generalization:

Theorem 1: Let M_1, M_2, M_3 be fin. dim. type 1 $U_q(\mathfrak{g})$-modules. Assume that f also satisfies the following multiplicativity formulas:

$$f(\lambda,\mu+\nu) = f(\lambda,\mu)f(\lambda,\nu)\ ,\quad f(\lambda+\nu,\mu) = f(\lambda,\mu)f(\nu,\mu)\quad \forall\,\lambda,\mu,\nu\in P$$

Then, $R^\vee := \Theta^f\circ\tau$ also satisfy the hexagon identities:

$$\begin{tikzcd}
 & M_1\otimes(M_3\otimes M_2) \arrow[r, "can"] & (M_1\otimes M_3)\otimes M_2 \arrow[dr, "R^\vee\otimes id"] & \\
M_1\otimes(M_2\otimes M_3) \arrow[ur, "id\otimes R^\vee"] \arrow[dr, "can"'] & & & (M_3\otimes M_1)\otimes M_2 \\
 & (M_1\otimes M_2)\otimes M_3 \arrow[r, "R^\vee_{12,3}"'] & M_3\otimes(M_1\otimes M_2) \arrow[ur, "can"'] &
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
 & (M_2\otimes M_1)\otimes M_3 \arrow[r, "can"] & M_2\otimes(M_1\otimes M_3) \arrow[dr, "id\otimes R^\vee"] & \\
(M_1\otimes M_2)\otimes M_3 \arrow[ur, "R^\vee\otimes id"] \arrow[dr, "can"'] & & & M_2\otimes(M_3\otimes M_1) \\
 & M_1\otimes(M_2\otimes M_3) \arrow[r, "R^\vee_{1,23}"'] & (M_2\otimes M_3)\otimes M_1 \arrow[ur, "can"'] &
\end{tikzcd}$$

► The proof is precisely the same as of [Lecture 15, Proposition 1] ◼

Exercise: Classify all $f: P\times P\to\mathbb{k}^*$ satisfying all the four relations

Remark: Theorem 2 translates into $R^\vee_{12}R^\vee_{23}R^\vee_{12} = R^\vee_{23}R^\vee_{12}R^\vee_{23}$. Indeed:

$LHS = \Theta^f_{12}\tau_{12}\Theta^f_{23}\tau_{23}\Theta^f_{12}\tau_{12} = \Theta^f_{12}\Theta^f_{13}\tau_{12}\tau_{23}\Theta^f_{12}\tau_{12} = \Theta^f_{12}\Theta^f_{13}\Theta^f_{23}\circ\tau_{12}\tau_{23}\tau_{12}$

$RHS = \Theta^f_{23}\tau_{23}\Theta^f_{12}\tau_{12}\Theta^f_{23}\tau_{23} = \Theta^f_{23}\Theta^f_{13}\tau_{23}\tau_{12}\Theta^f_{23}\tau_{23} = \Theta^f_{23}\Theta^f_{13}\Theta^f_{12}\circ\tau_{23}\tau_{12}\tau_{23}$

and finally $\tau_{12}\tau_{23}\tau_{12} = \tau_{23}\tau_{12}\tau_{23}$ (mapping $m_1\otimes m_2\otimes m_3 \mapsto m_3\otimes m_2\otimes m_1$)

If we now pick any type 1 f.d. $U_q(\mathfrak{g})$-module V and $r \geq 3$, then we get a family of linear operators $\underset{(1\leq i \leq r-1)}{R_i := \check{R}_{i,i+1}}$ acting on $V^{\otimes r}$. By above Remark, they satisfy $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$. Also $R_i R_j = R_j R_i$ whenever $|i-j| \geq 2$.

> <u>Corollary 1</u>: The endomorphisms $\{R_i\}_{i=1}^{r-1}$ define a representation of the braid group B_r on $V^{\otimes r}$.

<u>Remark</u>: Here, the (type A) <u>braid gp</u> B_r is abstractly defined by generators $\sigma_1, \ldots, \sigma_{r-1}$ subject to the relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2.$$

Note that we have a similar realization of the symmetric group S_r, for which we also add relations $\sigma_i^2 = id$ (here, σ_i represents a transposition $(i, i+1)$). We thus obtain:

$$\pi : B_r \twoheadrightarrow S_r$$

and $Ker(\pi)$ is called the <u>pure braid group</u>

<u>Example</u>: Evoking our computation from [Lecture 15, Remark] for $\mathfrak{g} = sl_2$, $M_1 = M_2 = L(1, +)$ and $f(q,q) = q^{-1}$, we have:

$$\Theta^f = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1}-q & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \Rightarrow R := \Theta^f \circ \tau = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

[<u>Exercise</u>: Verify $R^2 = (q^{-1}-q)R + 1 \Rightarrow (qR^{-1})^2 = (q^2-1)(qR^{-1}) + q^2$

Thus, combining this with the Corollary above, we obtain an action of <u>type A_{r-1} Hecke algebra</u> on $L(1,+)^{\otimes r}$ arising through the intertwiner R.

Lecture #29

Let G be the simply connected, connected, semisimple Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$.

Def: Let $\mathbb{k}_q[G]$ be the subspace of $U_q(\mathfrak{g})^*$ spanned by matrix coefficients $\{c_{f,m} \mid m \in M, f \in M^*$ where M - fin. dim. type 1 $U_q(\mathfrak{g})$-module$\}$.

Lemma 1: $\mathbb{k}_q[G]$ is actually a subalgebra (with unit)

▷ Let $m_1 \in M_1$, $f_1 \in M_1^*$, $m_2 \in M_2$, $f_2 \in M_2^*$. Then, we claim that

$$c_{f_1,m_1} \cdot c_{f_2,m_2} = c_{f_1 \otimes f_2, \underbrace{m_1 \otimes m_2}_{\in M_1 \otimes M_2}}$$

Indeed, for any $u \in U_q(\mathfrak{g})$:

$$c_{f_1\otimes f_2, m_1 \otimes m_2}(u) = (f_1 \otimes f_2)(u(m_1 \otimes m_2)) = \sum_{(u)} f_1(u'(m_1)) \cdot f_2(u''(m_2)) = \underbrace{\sum_{(u)} c_{f_1,m_1}(u') c_{f_2,m_2}(u'')}_{= (c_{f_1,m_1} \cdot c_{f_2,m_2})(u)}$$

Also, the unit is coming from taking $M = \mathbb{k}$ with the trivial (via counit) action, $m = 1 \in M$, $f = 1^* \in M^*$. Then: $c_{f,m} = \varepsilon$ which is identity for above product ▢

Exercise (Hwk 6): $\mathbb{k}_q[G]$ has a natural Hopf algebra structure.

Remark: For $\mathfrak{g} = \mathfrak{sl}_n$, the above construction ultimately recovers $SL_q(n)$ from [Homework 2, Problem 10]. The key reason for this is that $L(\omega_1)$ which is the q-deformation of $\mathfrak{sl}_n \curvearrowright \mathbb{C}^n$ generates all other f.d. type 1 $U_q(\mathfrak{g})$-modules under taking direct summands in $L(\omega_1)^{\otimes 2}$. And the matrix coeff-s of $L(\omega_1)$ are spanned by $\{c_{e_i^*, e_j} \mid 1 \le i,j \le n\}$

Exercise (Hwk 6): Identify $\mathbb{k}_q[SL_2]$ with $SL_{q^{-1}}(2)$ from Lecture 6.

Next time: Lusztig's braid group action and the PBW-type basis.