Lecture #30

Goal (today + 2-3 more classes): Construct braid gp action (of Lusztig) on $U_q(\mathfrak{g})$ which will yield a PBW-type basis for $U_q(\mathfrak{g})$ as well as more explicit f-la for (H).

Motivation: In the classical case, given any finite dim. module V over a simple Lie algebra $\mathfrak{g}$, its weights are invariant with respect to the Weyl group of $\mathfrak{g}$ generated by s_i (= i^{th} simple reflection). Moreso, an explicit isomorphism $V_\lambda \xrightarrow{\sim} V_{s_i(\lambda)}$ is given by

$$\boxed{\tilde{s}_i = \exp(e_i)\exp(-f_i)\exp(e_i) = \sum_{a,b,c\geq 0} \frac{e_i^a}{a!}\cdot\frac{(-f_i)^b}{b!}\cdot\frac{e_i^c}{c!}}$$

Applying this particularly to the adjoint action, we get
$\exp(\mathrm{ad}(e_i))\exp(-\mathrm{ad}(f_i))\exp(\mathrm{ad}(e_i)) = \mathrm{Ad}(\exp(e_i)\exp(-f_i)\exp(e_i)) = \mathrm{Ad}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$
(viewed as the image of $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ under the embedding $SL_2 \hookrightarrow G$).
Crucially important is that we get:

$$\boxed{\tilde{s}_i(Xv) = \tilde{s}_i(X)\,\tilde{s}_i(v) \quad \forall v\in V \quad \forall X\in\mathfrak{g} \text{ and hence also } X\in U(\mathfrak{g})}$$

Today: Introduce and study q-analogues of such operators.

Clearly, it suffices to introduce this first in the $\mathfrak{sl}_2$-case. In fact, Lusztig introduced 4 operators which we recall now.

Let V be a fin. dim. type 1 $U_q(\mathfrak{sl}_2)$-module, i.e. $V = \bigoplus_{m\in\mathbb{Z}} V_m$, $V_m = \{v \mid Kv = q^m\cdot v\}$
Define $T, T', {}^\omega T, {}^\omega T' \in \mathrm{End}(V)$ so that on any homogeneous $v\in V_m$:

$$\boxed{\begin{aligned} T(v) &:= \sum_{\substack{a,b,c\geq 0\\ -a+b-c=m}} (-1)^b q^{b-ac} E^{(a)}F^{(b)}E^{(c)}v, \quad & T'(v) &:= \sum_{\substack{a,b,c\geq 0\\ -a+b-c=m}} (-1)^b q^{-b+ac} E^{(a)}F^{(b)}E^{(c)}v \\ {}^\omega T(v) &:= \sum_{\substack{a,b,c\geq 0\\ a-b+c=m}} (-1)^b q^{b-ac} F^{(a)}E^{(b)}F^{(c)}v, \quad & {}^\omega T'(v) &:= \sum_{\substack{a,b,c\geq 0\\ a-b+c=m}} (-1)^b q^{-b+ac} F^{(a)}E^{(b)}F^{(c)}v \end{aligned}}$$

where we used the so-called divided powers:

$$\boxed{E^{(k)} := E^k/[k]_q! \quad \text{and} \quad F^{(k)} := F^k/[k]_q!}$$

Note: These f-las are obvious q-analogues of $\tilde{s}_i$ in the classical case, while the powers of $q^{\pm(b-ac)}$ are chosen to have nice properties.

Remark: a) As E, F act nilpotently on V, all the 4 sums are actually finite.

b) The additional requirement $\mp(a-b+c)=m$ guarantees that $T(v), T'(v), {}^{\omega}T(v), {}^{\omega}T'(v) \in V_{-m}$.

c) The difference for operators with/without primes is that in extra q-powers parameter q is inverted (i.e. q^{b-ac} vs q^{-b+ac})

d) Finally, the operators with ω are related to those without:
$${}^{\omega}T({}^{\omega}v) = {}^{\omega}(T(v)) \quad \text{and} \quad {}^{\omega}T'({}^{\omega}v) = {}^{\omega}(T'(v))$$
where ${}^{\omega}V$ denotes the twist of $U_q(\mathfrak{g})$-module V by Cartan involution ω, and ${}^{\omega}v$ denotes the vector v now viewed in ${}^{\omega}V$.

But all the above operators can be computed directly. To this end, we note that every $V \simeq \oplus L(n,+)$'s, hence suffices to treat $V = L(n,+)$. To simplify the formulas, we shall slightly rescale the basis for $L(n,+)$ from [Lecture 11, Proposition 1], i.e. we use the basis $\{v_i\}_{i=0}^{n}$ s.t.

$$K(v_i) = q^{n-2i} \cdot v_i$$
$$E(v_i) = \begin{cases} [n+1-i]_q \cdot v_{i-1}, & i>0 \\ 0, & i=0 \end{cases}$$
$$F(v_i) = \begin{cases} [i+1]_q \cdot v_{i+1}, & i<n \\ 0, & i=n \end{cases}$$

In particular, the divided powers act:
$$\boxed{F^{(r)} v_i = \begin{bmatrix} r+i \\ r \end{bmatrix} v_{i+r}, \quad E^{(r)} v_i = \begin{bmatrix} n+r-i \\ r \end{bmatrix} v_{i-r}}$$
with $v_j = 0$ if $j<0$ or $j>n$.

Lemma 1: The following formulas hold:

a) $T(v_i) = (-1)^{n-i} q^{(n-i)(i+1)} v_{n-i}$

b) $T'(v_i) = (-1)^{n-i} q^{-(n-i)(i+1)} v_{n-i}$

c) ${}^{\omega}T(v_i) = (-1)^{i} q^{i(n+1-i)} v_{n-i}$

d) ${}^{\omega}T'(v_i) = (-1)^{i} q^{-i(n+1-i)} v_{n-i}$

► The proof is straightforward, though cumbersome. Parts a)-b) reduce to:
$$\sum_{\substack{a,b,c \\ -a+b-c=n-2i}} (-1)^b q^{\pm(b-ac)} \cdot \begin{bmatrix} n+c-i \\ c \end{bmatrix} \begin{bmatrix} i-c+b \\ b \end{bmatrix} \begin{bmatrix} n+a-i-b+c \\ a \end{bmatrix} = (-1)^{n-i} \cdot q^{\pm(n-i)(i+1)}$$
which can be easily reduced to (after setting $j := n-i$):
$$\sum_{0 \le a,c \le i} (-1)^{b+j} q^{\pm(b-ac-j(i+1))} \begin{bmatrix} i \\ a \end{bmatrix} \begin{bmatrix} j+a \\ i-c \end{bmatrix} \begin{bmatrix} i \\ a \end{bmatrix} = 1 \quad \text{where } b = a+c+j-i$$
Exercise

Remark: The equalities from parts c–d) of Lemma 1 can be easily derived from a–b) using an obvious identification ${}^{\omega}L(n,+) \xrightarrow{\sim} L(n,+)$, ${}^{\omega}v_{n-i} \mapsto v_i$. Indeed:
${}^{\omega}T(v_i) = {}^{\omega}T({}^{\omega}v_{n-i}) = {}^{\omega}(T(v_{n-i})) \overset{a)}{=} (-1)^i q^{i(n-i+1)}\, {}^{\omega}v_i = (-1)^i q^{i(n+1-i)} v_{n-i} \Rightarrow c)$
and similarly d) can be derived from b)

Corollary 1: For any fin. dim. type 1 $U_q(\mathfrak{sl}_2)$-module V, we have:
a) $T^{-1} = {}^{\omega}T'$, $(T')^{-1} = {}^{\omega}T$ (in particular, all four are invertible)
b) $\forall v \in V_m$: ${}^{\omega}T(v) = (-q)^{-m} T(v)$, ${}^{\omega}T'(v) = (-q)^m T'(v)$

▶ Follows immediately from explicit f-las of Lemma 1 ∎

Lemma 2: For any fin. dim. type 1 $U(\mathfrak{sl}_2)$-module V, we have:
a) $T(Kv) = K^{-1}T(v)$, $T(Ev) = (-FK)T(v)$, $T(Fv) = (-K^{-1}E)T(v)$
b) $KT(v) = T(K^{-1}v)$, $FT(v) = T(-EKv)$, $ET(v) = T(-K^{-1}Fv)$

▶ a) It suffices to treat $V = L(n,+)$. The 1^{st} equality is clear. For the 2^{nd}:
$T(Ev_i) = [n+1-i]_q \cdot T(v_{i-1}) \overset{\text{Lemma 1}}{=\!=} [n+1-i]_q \cdot (-1)^{n-i+1} \cdot q^{(n-i+1)i} v_{n-i+1}$
$(-FK)T(v_i) \overset{\text{Lemma 1}}{=\!=} (-FK)((-1)^{n-i} q^{(n-i)(i+1)} v_{n-i}) = (-1)^{n-i+1} \cdot [n+1-i]_q \cdot q^{(n-i)(i+1)} \cdot q^{2i-n} v_{n-i+1}$
The 3^{d} equality is checked similarly.

b) Follows from a)

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Now we can clearly generalize all the above to any simple Lie alg. $\mathfrak{g}$.
Indeed, for any $i \in I$, we have $U_{q_i}(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{g})$ and thus any $U_q(\mathfrak{g})$-mod V can be also viewed as an $U_{q_i}(\mathfrak{sl}_2)$-module; get $\{T_i, T_i', {}^{\omega}T_i, {}^{\omega}T_i'\}_{i \in I} \subset \mathrm{End}(V)$.
All of these map $V_\lambda \longrightarrow V_{s_i(\lambda)} = \lambda - \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$.

$\boxed{\begin{array}{l}\text{\underline{Corollary 2}: a) } T_i, T_i', {}^{\omega}T_i, {}^{\omega}T_i' \text{ map isomorphically } V_\lambda \xrightarrow{\sim} V_{s_i(\lambda)} \\ \text{b) } T_i^{-1} = {}^{\omega}T_i', \ (T_i')^{-1} = {}^{\omega}T_i \\ \text{c) } {}^{\omega}T_i(v) = (-q_i)^{-m_i} T_i(v), \ {}^{\omega}T_i'(v) = (-q_i)^{m_i} T_i'(v) \ \text{ with } m_i = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} \\ \text{d) } T_i(E_i v) = (-F_iK_i)T_i(v), \ T_i(F_i v) = (-K_i^{-1}E_i)T_i(v) \\ \quad E_iT_i(v) = T_i(-K_i^{-1}E_i v), \ F_iT_i(v) = T_i(-E_iK_i v) \\ \text{e) } T_i(K_\mu v) = K_{s_i(\mu)}T_i(v)\end{array}}$

► a)-d) follow from $\mathfrak{sl}_2$-case

e) follows from $(\lambda,\mu) = (s_i(\lambda), s_i(\mu))$ ■

Note: If $i \neq j$ but $(\alpha_i, \alpha_j) = 0$, then E_j & F_j commute with E_i & F_i. Hence:

$$\boxed{T_i(E_j v) = E_j T_i(v), \quad T_i(F_j v) = F_j T_i(v) \text{ if } (\alpha_i,\alpha_j) = 0}$$

Up to now we know how $\{T_i\}$ commute with $\{E_j, F_j\}$ for $(\alpha_i, \alpha_j) \geq 0$.

The following result treats the remaining cases:

$\boxed{\begin{array}{l}\text{\underline{Proposition 1}: For any } i \neq j \text{ and a fin. dim. type 1 } U_q(\mathfrak{g})\text{-module } V, v \in V: \\ \quad T_i(E_j v) = \left(\mathrm{ad}(E_i^{(r)})E_j\right)T_i(v) \\ \quad T_i(F_j v) = \left(\sum_{k=0}^{r} (-1)^k q_i^k F_i^{(k)} F_j F_i^{(r-k)}\right)T_i(v) \qquad \text{with } r = -\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\end{array}}$

The proof of this result is based on basic properties of

$$\boxed{a(p) := \mathrm{ad}(E_i^{(p)})E_j, \quad p \geq 0}$$

Recall that $a(r+1) = 0$, hence, $a(p) = 0 \ \forall p > r$ by [Lecture 19, Lemma 3]