Lecture # 31

Last time we ended up with the following result:

Proposition 1: For any $i\neq j$ and a finite-dimensional type 1 $U_q(\mathfrak{g})$-module V, and $v\in V$:

$$T_i(E_j v) = ad(E_i^{(r)})E_j\ (T_i(v))$$
$$T_i(F_j v) = \Big(\sum_{k=0}^{r}(-1)^k q_i^k F_i^{(k)} F_j F_i^{(r-k)}\Big)(T_i(v))$$

with $r := -\dfrac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$

The proof of this result relies on the properties of the following elements:

$$\boxed{a(m) := ad(E_i^{(m)})E_j\ ,\quad m\geq 0}$$

First, recall that qSerre rel-n on $\{E_i\}_{i\in I}$ was exactly interpreted as: $a(r+1)=0$ (see [Lecture 19, Lemma 3]). Hence, $a(m)=0\ \forall m>r$. We also note:

$$\boxed{a(m) = \sum_{k=0}^{m}(-1)^k q_i^{k(m-1-r)} E_i^{(m-k)} E_j E_i^{(k)}}$$ by [Lecture 19, Lemma 2].

We now state a few technical lemmas on $\{a(m)\}$.

Lemma 1: $ad(F_i^{(s)})(a(m)) = \begin{cases} \begin{bmatrix} r+s-m \\ s\end{bmatrix}_{q_i}\cdot a(m-s) & , \text{ if } s\leq m \\ 0 & , \text{ if } s>m\end{cases}$

▸ As $ad(F_i^{(s)}) = \dfrac{1}{[s]_{q_i}!}ad(F_i)^s$ and $ad(F_i)E_j=0$ for $i\neq j$, it suffices to prove $s=1$ case.

But recall that $ad(F_i)u = [F_i,u]K_i$ by [Lecture 19, Lemma 1]. Hence:

$$[F_i, a(m)] = [F_i, \textstyle\sum_{k=0}^{m}(-1)^k q_i^{k(m-1-r)} E_i^{(m-k)} E_j E_i^{(k)}] \overset{\text{[Lecture 9, Lemma 1]}}{=\!=}$$

$$= \sum_{k=0}^{m}(-1)^k q_i^{k(m-1-r)}\Big(-E_i^{(m-k-1)}[K_i;m-k-1]E_j E_i^{(k)} - E_i^{(m-k)}E_j E_i^{(k-1)}[K_i;k-1]\Big) =$$

$$= \sum_{k=0}^{m-1}(-1)^{k-1} q_i^{k(m-1-r)} E_i^{(m-k-1)}\cdot \frac{K_i q_i^{m-k-1} - K_i^{-1}q_i^{-m+k+1}}{q_i-q_i^{-1}} E_j E_i^{(k)} +$$

$$\sum_{k=0}^{m-1}(-1)^k q_i^{(k+1)(m-1-r)} E_i^{(m-k-1)}E_j E_i^{(k)}\cdot\frac{K_i q_i^k - K_i^{-1}q_i^{-k}}{q_i-q_i^{-1}} =$$

here, we changed $k-1 \rightsquigarrow k$

we pass $[K_i;m-k-1]$ to the rightmost side

$$= \sum_{k=0}^{m-1}(-1)^k E_i^{(m-k-1)}E_j E_i^{(k)}\Big(q_i^{(k+1)(m-r-1)}\cdot\frac{K_i q_i^k - K_i^{-1}q_i^{-k}}{q_i-q_i^{-1}} - q_i^{k(m-r-1)}\cdot\frac{K_i q_i^{m+k-r-1} - K_i^{-1}q_i^{-m-k+r+1}}{q_i-q_i^{-1}}\Big)$$

$$= [r+1-m]_{q_i}\cdot a(m-1)\cdot K_i^{-1}$$

Therefore: $ad(F_i)(a(m)) = [F_i, a(m)]\cdot K_i = [r+1-m]_{q_i}\cdot a(m-1)$ ■

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Lemma 2: For any $m, p \geq 0$, we have:

a) $a(m) E_i^{(p)} = \sum_{k=0}^{p} (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q_i} \cdot q_i^{p(r-2m)-k(p-1)} E_i^{(p-k)} a(m+k)$

b) $a(m) F_i^{(p)} = \sum_{k=0}^{p} (-1)^k \begin{bmatrix} r-m+k \\ k \end{bmatrix}_{q_i} \cdot q_i^{k(p-1)} F_i^{(p-k)} a(m-k) K_i^{-k}$

▸ We shall first check both formulas for $p=1$ (the case $p=0$ is obvious)

- $ad(E_i) a(m) = ad(E_i) ad(E_i^{(m)}) E_j = [m+1]_{q_i} \cdot ad(E_i^{(m+1)}) E_j = [m+1]_{q_i} \cdot a(m+1)$
  || [Lecture 19, Lemma 1]
  $E_i \cdot a(m) - K_i a(m) K_i^{-1} \cdot a(m) = E_i \cdot a(m) - q_i^{2m-r} \cdot a(m) E_i$
  $\Rightarrow$

$\Rightarrow \boxed{a(m) E_i = q_i^{r-2m} E_i \cdot a(m) - q_i^{r-2m} [m+1]_{q_i} \cdot a(m+1)}$ which agrees with a)

- $ad(F_i) a(m) = [r+1-m]_{q_i} \cdot a(m-1)$ by Lemma 1
  ||
  $(F_i a(m) - a(m) F_i) K_i$
  $\Rightarrow$

$\Rightarrow \boxed{a(m) F_i = F_i a(m) - [r+1-m]_{q_i} \cdot a(m-1) \cdot K_i^{-1}}$ which agrees with b)

One can now deduce the general case by induction on p

[**Exercise**: Prove Lemma 2 by induction on p.

In particular, combining Lemma 2a) for $m=r$ with aforementioned $a(r+1)=0$ we get:

Corollary 1: $a(r) E_i^{(p)} = q_i^{-pr} E_i^{(p)} a(r)$

[**Exercise**: Prove the first equality of Proposition 1

Hint: One evaluates $a(r) T_i(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b-c=s=\frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}}} (-1)^b q_i^{b-ac} a(r) E_i^{(a)} F_i^{(b)} E_i^{(c)} v$ where $v \in V_\lambda$ by pulling $a(r)$ to the rightmost term with the help of Cor 1 and Lemma 2. When doing so, one gets a sum of terms of the form:

$\lfloor \text{coeff} \rfloor \, E_i^{(a)} F_i^{(b-p)} E_i^{(c-t)} a(r-p+t) v$, and the rest is to show that the overall sum simplifies to $T_i(E_j v) = T_i(a(0) v)$

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With Proposition 1 at hands, together with [Lecture 30, Corollary 2e)], we get:

Theorem 1: a) For any $i \in I$, $u \in U_q(\mathfrak{g})$ $\exists!\, u' \in U_q(\mathfrak{g})$ s.t.

$$\boxed{T_i(uv) = u'\,T_i(v)}$$

for any f.dim. type 1 $U_q(\mathfrak{g})$-module V and any $v \in V$.

b) Moreover, the assignment $u \mapsto u'$ is an automorphism of $U_q(\mathfrak{g})$

▶ a) By Prop 1 and [Lect 30, Cor 2e)], such u' exists when u is any of the generators $\{E_j, F_j, K_j^{\pm 1}\}$. Hence, u' exists $\forall u$.

Uniqueness of u' follows from the fact that an elt of $U_q(\mathfrak{g})$ acting by zero on all f.d. dim. type 1 $U_q(\mathfrak{g})$-modules is zero, which is [Hwk 4, Problem 4].

b) Clearly $(u_1+u_2)' = u_1' + u_2'$. The equality $(u_1u_2)' = u_1'u_2'$ follows from:

$$\underbrace{T_i(u_1u_2v)}_{\substack{\shortparallel \\ u_1'T_i(u_2v)\,=\,u_1'u_2'T_i(v)}} = (u_1u_2)'\,T_i(v)$$

Thus, $u \mapsto u'$ defines an algebra homomorphism $U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$.

Its injectivity follows as above: if $u_1' = u_2' \Rightarrow T_i((u_1-u_2)v) = 0 \overset{T_i\text{-invertible}}{\Longrightarrow} u_1 - u_2$ acts by zero on all f.dim type 1 modules $\Rightarrow u_1 = u_2$.

Finally, the surjectivity is straightforward are requires to find explicit $u \in U_q(\mathfrak{g})$ s.t. u' is any of the generators $\{E_j, F_j, K_j^{\pm 1}\}$. We leave this as an exercise.

[Exercise: Prove surjectivity of $u \mapsto u'$

Hint: Consider an algebra homomorphism ${}^{\sigma}\mathrm{ad}: U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbb{k}}(U_q(\mathfrak{g}))$
$$x \mapsto \sigma \circ \mathrm{ad}(x) \circ \sigma : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$$

a) Prove $({}^{\sigma}\mathrm{ad}(E_i)u)' = \mathrm{ad}(F_i)u'$

b) Prove $({}^{\sigma}\mathrm{ad}(F_i)u)' = \mathrm{ad}(E_i)u'$

c) Combine a)-b) with Lemma 2 to find u s.t. $u' = E_j$ or F_j

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We shall denote the resulting automorphisms of $U_q(\mathfrak{g})$ by T_i again, so that

$$\boxed{T_i(uv) = T_i(u)\,(T_i(v)) \quad \forall u\in U_q(\mathfrak{g}),\ \forall v\in V \text{ - f. dim. type 1}}$$

Evoking explicit formulas of Proposition 1 and [Lecture 30, Corollary 2d-e)] we obtain the following explicit formulas:

$$\boxed{\begin{aligned} T_i:\ & E_i \mapsto -F_iK_i, \quad F_i \mapsto -K_i^{-1}E_i \\ & E_j \mapsto \sum_{k=0}^{r} (-1)^k q_i^{-k} E_i^{(r-k)} E_j E_i^{(k)} \quad \text{for } j\neq i \quad \text{with } r = -\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \\ & F_j \mapsto \sum_{k=0}^{r} (-1)^k q_i^{k} F_i^{(k)} F_j F_i^{(r-k)} \quad \text{for } j\neq i \\ & K_\mu \mapsto K_{s_i(\mu)} \end{aligned}}$$

Evoking the Hint from the bottom of p.3, one also get f-las for T_i^{-1}:

$$\boxed{\begin{aligned} T_i^{-1}:\ & E_i \mapsto -K_i^{-1}F_i, \quad F_i \mapsto -E_iK_i \\ & E_j \mapsto \sum_{k=0}^{r} (-1)^k q_i^{-k} E_i^{(k)} E_j E_i^{(r-k)} \quad \text{for } j\neq i \\ & F_j \mapsto \sum_{k=0}^{r} (-1)^k q_i^{k} F_i^{(r-k)} F_j F_i^{(k)} \quad \text{for } j\neq i \\ & K_\mu \mapsto K_{s_i(\mu)} \end{aligned}}$$

We shall now conclude with the discussion of how $\{T_i\}_{i\in I}$ commute among themselves. We start with the simplest case:

Lemma 3: If $(\alpha_i,\alpha_j)=0$, then $T_iT_j=T_jT_i$ (as automorphisms of $U_q(\mathfrak{g})$ as well as endomorphisms of V)

► If $(\alpha_i,\alpha_j)=0 \Rightarrow \{E_i,F_i\}$ commute with $\{E_j,F_j\} \Rightarrow \forall$ f. dim. type 1 module V and any $v\in V$, we have $T_iT_j(v) = T_jT_i(v)$. But then $\forall u\in U_q(\mathfrak{g})$:

$$\begin{aligned} T_iT_j(uv) &= (T_iT_j(u))\,\underbrace{(T_iT_j(v))}_{\shortparallel} \\ \shortparallel \quad & \\ T_jT_i(uv) &= (T_jT_i(u))\,\overbrace{(T_jT_i(v))} \end{aligned} \quad \Rightarrow T_iT_j(u) = T_jT_i(u) \Rightarrow T_iT_j = T_jT_i \text{ as automorphisms of } U_q(\mathfrak{g}))$$

■

Next time we will see that $\forall i\neq j$, we actually have $\underbrace{T_iT_jT_i\ldots}_{N} = \underbrace{T_jT_iT_j\ldots}_{N}$