Lecture # 32

Last time we concluded that there are same-named automorphisms T_i of $U_q(\mathfrak{g})$ s.t.

$$\boxed{T_i(uv) = T_i(u)\,T_i(v)} \quad \forall u \in U_q(\mathfrak{g}) \quad \forall v \in V \leftarrow \text{fin. dim. type 1 module}$$

Their key property is:

Theorem 1: Let N be the order of $s_i s_j$ in the Weyl group ($i \neq j$). Then:

$$\underbrace{T_i T_j T_i \ldots}_{N} = \underbrace{T_j T_i T_j \ldots}_{N} \quad \leftarrow \text{equality of automorphisms of } U_q(\mathfrak{g}).$$

(Remark: In fact, the same equality also holds for any V (both sides being in $\mathrm{End}(V)$))

Def: Given a simple Lie algebra $\mathfrak{g}$ with $\Pi = \{\alpha_i\}_{i \in I}$ - set of simple roots and W = Weyl gp, one defines the braid group (of type $\mathfrak{g}$) as the group generated by $\{\sigma_i\}_{i \in I}$ subject only to the following rel-s:

$$\underbrace{\sigma_i \sigma_j \sigma_i \ldots}_{N} = \underbrace{\sigma_j \sigma_i \sigma_j \ldots}_{N} \quad \text{with } N = \text{order of } s_i s_j \text{ in } W$$

Remark: a) When $\mathfrak{g} = \mathfrak{sl}_n$, one recovers the standard braid group we encountered in Lecture 29 (in a rather different discussion of $U_q(\mathfrak{g}) \curvearrowright V, V^{\otimes n}$)

b) The quotient of the braid gp by $\sigma_i^2 = 1$ $\forall i$ recovers W = Weyl gp.

c) Thus Theorem 1 above means one has a braid gp action on $U_q(\mathfrak{g})$.

Proof of Theorem 1

There are 4 cases for possible values of N, and we shall check each separately. We shall use a_{ij} to denote (i,j)-entry of the Cartan matrix

Case 1: $(\alpha_i, \alpha_j) = 0 \Leftrightarrow a_{ij} = 0 = a_{ji} \Leftrightarrow N = 2$

But the corresponding equality $T_i T_j = T_j T_i$ was already established in Lect 31. ✓

Case 2: $\underbrace{a_{ij} = -1 = a_{ji}}_{\text{also implies } q_i = q_j} \Leftrightarrow N = 3$

We need to check $T_i T_j T_i = T_j T_i T_j$. It suffices to check this on the generators $\{E_k, F_k, K_k^{\pm 1}\}$. The action on Cartan K_μ agrees as we have equality $s_i s_j s_i = s_j s_i s_j$ in W. Also note the symmetry $i \leftrightarrow j$.

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(Continuation of the proof)

Evoking explicit formulas from p.4 of Lecture 31, we have:

$$\boxed{T_i(E_j) = E_iE_j - q_i^{-1}E_jE_i = T_j^{-1}(E_i)\ ,\quad T_i(F_j) = F_jF_i - q_iF_iF_j = T_j^{-1}(F_i)}$$

Hence:

- $$\left.\begin{array}{l} T_iT_jT_i(E_j) = T_i(E_i) = -F_iK_i \\ T_jT_iT_j(E_j) = T_jT_i(-F_jK_j) = -F_iK_{s_js_i(\alpha_j)} = -F_iK_i \end{array}\right\} \Rightarrow T_iT_jT_i(E_j) = T_jT_iT_j(E_j)$$
- $$\left.\begin{array}{l} T_iT_jT_i(F_j) = T_i(F_i) = -K_i^{-1}E_i \\ T_jT_iT_j(F_j) = T_jT_i(-K_j^{-1}E_j) = -K^{-1}_{s_js_i(\alpha_j)}E_i = -K_i^{-1}E_i \end{array}\right\} \Rightarrow T_iT_jT_i(F_j) = T_jT_iT_j(F_j)$$
- Due to $i \leftrightarrow j$ symmetry, also see that images of E_i, F_i agree.
- Finally, we need to verify that the images of E_k and F_k agree ($k \neq i,j$)

  But: $\mathfrak{g}$-simple $\Rightarrow$ no cycles in Dynkin diagram $\overset{a_{ij}\neq 0}{\Longrightarrow}$ $a_{ik}=0$ or $a_{jk}=0$.

  Due to $i \leftrightarrow j$ symmetry, we shall assume that $(\alpha_j,\alpha_k)=0 \Leftrightarrow a_{jk}=0$.

  Then: $\{E_j, F_j\}$ commute with $\{E_k, F_k\}$, and subsequently $T_j(E_k)=E_k,\ T_j(F_k)=F_k$.

Hence, $T_iT_jT_i(E_k) = T_jT_iT_j(E_k) \Leftrightarrow T_i(T_jT_i(E_k)) = T_jT_i(E_k)$

Instead of brute force computations, we note that it suffices to prove that $T_jT_i(E_k)$ commutes with both E_i & F_i for the same reason as we treated Case 1 in the end of Lecture 31

[More precisely, on any fin. dim. type 1 module V, we have then
$T_i(T_jT_i(E_k)v) = T_jT_i(E_k)T_i(v) \Rightarrow T_i(T_jT_i(E_k)) = T_jT_i(E_k)$]

But: $[E_j, E_k]=0 \overset{T_jT_i}{\Longrightarrow} [\underbrace{T_jT_i(E_j)}_{=E_i}, T_jT_i(E_k)]=0 \Rightarrow [E_i, T_jT_i(E_k)]=0$

$[F_j, E_k]=0 \overset{T_jT_i}{\Longrightarrow} [T_jT_i(F_j), T_jT_i(E_k)]=0 \Rightarrow [F_i, T_jT_i(E_k)]=0$ ✓

- Case 3: $a_{ji}=-2$ & $\underbrace{a_{ij}=-1}_{\text{in particular } q_i=q_j^2} \longleftrightarrow N=4$

[Exercise: Verify $T_iT_jT_iT_j = T_jT_iT_jT_i$]

- Case 4: $a_{ji}=-3$ & $a_{ij}=-1 \longleftrightarrow N=6$

  This long computation is unpleasant $\rightarrow$ interested reader to look in Lusztig's book.

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We shall now recall some basic results / notions on Weyl groups.

- Given any $w \in W$, we call decomposition
$$w = s_{i_1} s_{i_2} \ldots s_{i_\ell} \quad \text{into the product of simple reflections}$$
reduced if ℓ takes the minimal possible value. This value $\ell = \ell(w)$ is called the length of w. The reduced decomposition is not unique, but:

Fact 1: Given two reduced decompositions of the same element $w \in W$, one can get one from another by a sequence of elementary moves
$$\underbrace{s_i s_j s_i \ldots}_{N} \overset{\text{replace}}{\rightsquigarrow} \underbrace{s_j s_i s_j \ldots}_{N} \quad \text{with } N = \text{order of } s_i s_j \text{ in } W.$$

Combining this Fact with Theorem 1, we immediately obtain a family of automorphisms $\{T_w\}_{w \in W} \in \mathrm{Aut}(U_q(\mathfrak{g}))$, namely
$$\boxed{T_w := T_{i_1} \ldots T_{i_\ell} \quad \text{for any reduced decomposition } w = s_{i_1} \ldots s_{i_\ell}}$$

Remark: Since $\{T_i\}_{i \in I}$ also satisfy braid rel-s when viewed as elts in $\mathrm{End}(V)$, we also get a family $\{T_w\}_{w \in W}$ of endomorphisms of V.

Note: a) $T_{w_1 w_2} = T_{w_1} T_{w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ BUT not in general, e.g. $T_{s_i} T_{s_i} = T_i^2 \neq \mathrm{id}$.

b) $T_w(K_\mu) = K_{w(\mu)} \; \forall \mu$

c) $T_w^{-1} = \sigma \circ T_{w^{-1}} \circ \sigma$

Exercise

- Our next goal is to construct a PBW-type basis of $U_q^{\pm}$ using these T_w. To do so, as well as to motivate the construction, we recall:

Fact 2: For any reduced decomposition $w = s_{i_1} s_{i_2} \ldots s_{i_\ell}$, we have
$$\boxed{\{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} s_{i_2} \ldots s_{i_{\ell-1}}(\alpha_{i_\ell})\} = \{\gamma \in \Delta^+ \mid w^{-1}(\gamma) \in \Delta^-\}}$$
In particular, the length $\ell(w) = \#\{\gamma \in \Delta^+ \mid w^{-1}(\gamma) \notin \Delta^+\}$

the right-hand side is indep. of reduced decomp.

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The above result motivates its q-version discovered by Lusztig:

Theorem 2: For any $w \in W$ pick a reduced decomp. $w = s_{i_1} \ldots s_{i_\ell}$ and consider
$$\{T_{i_1} T_{i_2} \ldots T_{i_{\ell-1}} (E_{i_\ell})^{a_\ell} \cdot \ldots \cdot T_{i_1}(E_{i_2})^{a_2} \cdot E_{i_1}^{a_1} \mid a_1, \ldots, a_\ell \in \mathbb{Z}_{\geq 0}\}$$
a) These ordered products are linearly independent

b) The subspace these elts span is independent of a reduced decompos.

$\Downarrow$ Notation: $U^+[w]$ will denote this subspace

While we shall prove this result next time, we will now show that all these elements are indeed in U_q^+. We start from simple lemma:

Lemma 1: Let $w \in \underbrace{\langle s_i, s_j \rangle}_{\text{subgp generated by } s_i \& s_j} \leq W$ for $i \neq j$, and $w(\alpha_i) \in \Delta^+$. Then:

a) $T_w(E_i) \in \langle E_i, E_j \rangle$ - subalgebra of U_q^+ generated by E_i, E_j

b) $T_w(E_i) = T_{w(\alpha_i)}$ if $w(\alpha_i)$ - simple positive root

▶ This is a rank 2 statement and can be checked case-by-case, treating the same four cases as in our proof of Theorem 1.

Case 1: $N=2$ (so that $a_{ij} = 0 = a_{ji}$)

Then $W = \{1, s_i, s_j, s_i s_j = s_j s_i\}$. The only $w \in W$ s.t. $w(\alpha_i) \in \Delta^+$ are $w \in \{1, s_j\}$. But for both $w(\alpha_i) = \alpha_i$ and $T_w(E_i) = E_i$. ✓

Case 2: $N=3$ (so that $a_{ij} = -1 = a_{ji}$)

Then $W = \{1, s_i, s_j, s_i s_j, s_j s_i, s_i s_j s_i = s_j s_i s_j\}$. Easy to check that w s.t. $w(\alpha_i) \in \Delta^+$ are $w \in \{1, s_j, s_i s_j\}$. But $T_1(E_i) = E_i$, $T_j(E_i) = E_j E_i - q_i^{-1} E_i E_j$, and $T_i T_j(E_i) = E_j$ as already shown before. ✓

Case 4: $N=4$

[Exercise

Case 5: $N=6$ $\to$ direct unpleasant calculation

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With the previous lemma at hand, we can now prove the general case:

Proposition 1: Let $w \in W (= \text{Weyl gp of } \mathfrak{g})$ and $i \in I$ be such that $w(\alpha_i) \in \Delta^+$.

a) $T_w(E_i) \in U_q^+$

b) $T_w(E_i) = E_{w(\alpha_i)}$ if $w(\alpha_i)$ - simple root $\leftarrow$ here, we identify $E_j \leftrightarrow E_{\alpha_j}$

The proof is based on another basic fact on Weyl gps.

Fact 3: Fix any subset J of I (= the set of simple roots). We define the subgp $\boxed{W_J := \langle s_j \rangle_{j \in J}}$ as generated by all simple reflections s_j with $j \in J$.
We also define the set $\boxed{W^J := \{w \in W \mid \ell(ws_j) > \ell(w) \ \forall j \in J\}}$.
Then any $w \in W$ admits a unique decomposition
$$\boxed{w = w' \cdot w'' \text{ with } w' \in W^J, w'' \in W_J}$$
Moreover, $\ell(w) = \ell(w') + \ell(w'')$ and w' is the shortest elt in $w \cdot W_J$.

Proof of Proposition 1

- If $w=1$, the result is obvious. If $w \neq 1$, then $\exists j \in I$ s.t. $\overset{\text{negative root}}{w(\alpha_j)} \notin \Delta^+$. We apply the above Fact 3 to $J = \{i, j\} \subseteq I$, to get $w = w'w''$ as above.
Evoking the def-n of W^J, the obvious equality $\ell(u) = \ell(u^{-1})$ $\forall u \in W$, and the combinatorial meaning of $\ell(u)$ from Fact 2, we get: $w'(\alpha_i), w'(\alpha_j) \in \Delta^+$.

As $w'' \in \langle s_i, s_j \rangle$, we see that each of $w''(\alpha_i), w''(\alpha_j)$ is of the form $a\alpha_i + b\alpha_j$ with $a, b \in \mathbb{Z}_{\geq 0}$ or $a, b \in \mathbb{Z}_{\leq 0}$. As $w(\alpha_i) \in \Delta^+, w(\alpha_j) \in \Delta^-, w'(\alpha_i) \in \Delta^+, w'(\alpha_j) \in \Delta^+$ we conclude: $\boxed{w''(\alpha_i) \in \Delta^+, w''(\alpha_j) \in \Delta^-}$

a) By Lemma 1, $T_{w''}(E_i) \in \langle E_i, E_j \rangle$ (this is rank 2 calculation). As $\ell(w') < \ell(w)$ and $w'(\alpha_i), w'(\alpha_j) \in \Delta^+$, we have $T_{w'}(E_i), T_{w'}(E_j) \in U_q^+$ by induction on length. Finally, $T_w = T_{w'} T_{w''}$ as $\ell(w) = \ell(w') + \ell(w'')$, hence, $T_w(E_i) \in U_q^+$.

b) If $w(\alpha_i)$ - simple root, then $w''(\alpha_i)$ - simple root (hence α_i or α_j). If not then $w''(\alpha_i) = a\alpha_i + b\alpha_j$ with $a, b > 0 \Rightarrow w(\alpha_i) = a\underbrace{w'(\alpha_i)}_{\in \Delta^+} + b\underbrace{w'(\alpha_j)}_{\in \Delta^+} \Rightarrow \Downarrow$
The result again follows from Lemma 1 and induction