Lecture #33

Goals for today: 1) Prove Thm 2 of Lecture 32

2) Use it to derive bases of $U_q^{\geq}$.

Proof of [Lecture 32, Theorem 2]

a) Want to prove

(1) $$\boxed{\{T_{i_1}\dots T_{i_{\ell-1}}(E_{i_\ell})^{a_\ell}\cdot\ \_\_\_\ \cdot T_{i_1}(E_{i_2})^{a_2}\cdot E_{i_1}^{a_1}\,|\,a_1,\dots,a_\ell\in\mathbb{Z}_{\geq 0}\}\text{ - lin. indep}}$$

for any reduced decomposition $w=s_{i_1}s_{i_2}\dots s_{i_\ell}$. The proof is by induction on $l(w)$. If $l(w)=0\Rightarrow w=1\Rightarrow$ nothing to check. Otherwise, write $w=s_{i_1}(s_{i_2}\dots s_{i_\ell})=s_{i_1}w'$. Note that $w'=s_{i_2}\dots s_{i_\ell}$ is reduced, hence, by induction hypothesis the elements

(2) $$\boxed{\{T_{i_2}\dots T_{i_{\ell-1}}(E_{i_\ell})^{a_\ell}\cdot\ \_\_\_\ \cdot T_{i_2}(E_{i_3})^{a_3}\cdot E_{i_2}^{a_2}\,|\,a_2,\dots,a_\ell\in\mathbb{Z}_{\geq 0}\}\text{ - lin. indep.}}$$

On the other hand, applying $T_{i_1}^{-1}$ to any linear dependence:

$$\sum_{a_1,\dots,a_\ell} c_{a_1,\dots,a_\ell}\, T_{i_1}\dots T_{i_{\ell-1}}(E_{i_\ell})^{a_\ell}\cdot\ \_\_\ \cdot E_{i_1}^{a_1}=0,$$

we get

$$\boxed{\sum_{a_1,\dots,a_\ell} c_{a_1,\dots,a_\ell}\cdot\left(T_{i_2}\dots T_{i_{\ell-1}}(E_{i_\ell})^{a_\ell}\cdot\ \_\_\ \cdot E_{i_2}^{a_2}\right)\cdot(-K_{i_1}^{-1}F_{i_1})^{a_1}=0}$$

But $U_q^+\otimes U_q^{\leq}\xrightarrow[\text{mult}]{\sim}U_q(\mathfrak{g})$ - isomorphism of vector spaces (by triangular decomp). Thus, combining lin. indep. of (2) with lin. independence of $\{(-K_{i_1}^{-1}F_{i_1})^{a_1}\}_{a_1\geq 0}$ (for degree reasons), we immediately conclude that all $c_{a_1,\dots,a_\ell}=0$. ✓

b) Want to prove that the span of the aforementioned products is independent of a reduced decomposition of w. But, evoking Fact 1 from Lecture 32, it suffices to show that this span does not change under a single elementary move:

$$\boxed{w=w_1\underbrace{s_is_js_i\dots}_{N}w_2\ \xrightarrow[\text{move}]{\text{elementary}}\ w=w_1\underbrace{s_js_is_j\dots}_{N}w_2}$$

$N=$ order of s_is_j in W

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▶ (Continuation)

However, it's easy to see that this essentially reduces to the rank 2 case.
(Indeed, if $w_1 = s_{i_1} \dots s_{i_p}$, $w_2 = s_{i_{p+1}} \dots s_{i_t}$, so that $\ell = N+t$, then the first $t-p$ and the last p terms in the sequence $T_{i_1} \dots T_{i_{r-1}}(E_{i_r})$ are independent of the above elementary move)

Thus, the result follows from the following simple lemma:

> **Lemma 1**: If $i \neq j$ and $\tilde{w} = w_0^J = \underbrace{s_i s_j s_i \dots}_{N} = \underbrace{s_j s_i s_j \dots}_{N}$ is the longest elt in $\langle s_i, s_j \rangle = W^J$ with $J = \{i, j\}$, then the span of the corresponding products is $\langle E_i, E_j \rangle$ - subalgebra generated by E_i, E_j.

▷ This is a rank 2 computation that can be checked case-by-case!

Case 1: $N=2$ $(a_{ij} = 0 = a_{ji})$

Then $\tilde{w} = s_i s_j \Rightarrow T_i(E_j)^{a_2} \cdot E_i^{a_1} = E_j^{a_2} \cdot E_i^{a_1} \Rightarrow$ they span $\langle E_i, E_j \rangle$ as $[E_i, E_j] = 0$

Case 2: $N=3$ $(a_{ij} = -1 = a_{ji})$

Then $\tilde{w} = s_i s_j s_i \Rightarrow T_i T_j (E_i)^{a_3} \cdot T_i(E_j)^{a_2} \cdot E_i^{a_1} = E_j^{a_3} \cdot (E_i E_j - q_i^{-1} E_j E_i) \cdot E_i^{a_1}$.

Want: prove $\boxed{\mathrm{Span}\{E_j^{a_3} (E_i E_j - q_i^{-1} E_j E_i)^{a_2} \cdot E_i^{a_1}\} = \langle E_i, E_j \rangle}$

We clearly have $\subseteq$. To prove $=$, it suffices to check that this span is stable under left multiplication by E_i, E_j (as it contains 1).

The case of E_j is obvious: $E_j \cdot E_j^{a_3} (E_i E_j - q_i^{-1} E_j E_i)^{a_2} \cdot E_i^{a_1} = E_j^{a_3+1} (E_i E_j - q_i^{-1} E_j E_i)^{a_2} E_i^{a_1}$

Thus, it remains to rewrite $E_i \cdot E_j^{a_3} (E_i E_j - q_i^{-1} E_j E_i)^{a_2} \cdot E_i^{a_1}$ as a linear combination of such "ordered products".

We thus need to "move" E_i to the right of $E_j^{a_3}$ and then to the right of $E_i E_j - q_i^{-1} E_j E_i$. The latter is easy:

$$\underbrace{E_i^2 E_j - (q_i + q_i^{-1}) E_i E_j E_i + E_j E_i^2 = 0}_{q\text{Serre rel-n}} \Rightarrow \boxed{E_i (E_i E_j - q_i^{-1} E_j E_i) = q_i (E_i E_j - q_i^{-1} E_j E_i) E_i}$$

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▷ (Continuation)

It remains to move E_i to the right of $E_j^{a_3}$. We illustrate this now:

- if $a_3 = 1$, then

$$E_iE_j = q_i^{-1}E_jE_i + (E_iE_j - q_i^{-1}E_jE_i)$$

- if $a_3 = 2$, then

$$E_iE_j^2 \underset{q\text{Serre}}{=\!=} (q_j+q_j^{-1})E_jE_iE_j - E_j^2E_i \underset{q_i=q_j}{=\!=} q_i^{-2}\underbrace{E_j^2E_i}_{} + (q_i+q_i^{-1})\underbrace{E_j(E_iE_j - q_i^{-1}E_jE_i)}_{}$$

both ordered as wanted

- if $a_3 = 3$, then

$$E_iE_j^3 = E_iE_j^2 \cdot E_j = \left(q_i^{-2}E_j^2E_i + (q_i+q_i^{-1})E_j(E_iE_j - q_i^{-1}E_jE_i)\right)E_j =$$

$$= q_i^{-2}E_j^2(q_i^{-1}E_jE_i + (E_iE_j - q_i^{-1}E_jE_i)) + (q_i+q_i^{-1})E_j\underbrace{(E_iE_j - q_i^{-1}E_jE_i)E_j}_{\overset{q\text{Serre}}{=\!=} q_iE_j(E_iE_j - q_i^{-1}E_jE_i)}$$

$$= q_i^{-3}E_j^3E_i + \underbrace{(q_i^2 + 1 + q_i^{-2})}_{=[3]_{q_i}}E_j^2(E_iE_j - q_i^{-1}E_jE_i)$$

Easy to prove by induction:

$$\boxed{E_iE_j^{a_3} = q_i^{-a_3}E_j^{a_3}E_i + [a_3]_{q_i}E_j^{a_3-1}(E_iE_j - q_i^{-1}E_jE_i)}$$

as wanted. This completes Case 2 in the proof of Lemma 1

Case 3: $N=4$

[Exercise

Case 4: $N=6$ $\longrightarrow$ as always, this is the most unpleasant computation we skip $\square$

This completes our proof of Theorem 2 from Lecture 32 ∎

We denote the span considered above by $U^+[w]$. Applying the Cartan involution ω of $U_q(\mathfrak{g})$, we define $U^-[w] := \omega(U^+[w]) \subseteq U_q^-$

(Remark: In fact, $U^\pm[w]$ are not just subspaces but also are subalgebras

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The following simple lemma is needed for our main result.

Lemma 2: If $w \in W$, $i \in I$ satisfy $w^{-1}(\alpha_i) \in \Delta^-$, then $U^+[w] \cdot E_i \subseteq U^+[w]$

▶ It suffices to show that w admits a reduced decomposition
$$w = s_{i_1} s_{i_2} \ldots s_{i_\ell} \quad \text{with } i_1 = i$$
since all the "ordered" products are then mapped to "ordered" products under the right multiplication by E_i. To this end, it suffices to show
$$\ell(s_i w) < \ell(w)$$
as then picking any reduced decomposition $s_i w = s_{i_2} \ldots s_{i_\ell}$ yields the claim.

But: $s_i : \alpha_i \mapsto -\alpha_i$, $\Delta^+ \setminus \{\alpha_i\} \circlearrowleft$ $\Rightarrow$ $\ell(s_i w) = \#\{\gamma \in \Delta^+ \mid (s_i w)^{-1} \gamma \in \Delta^-\}$
$$\shortparallel$$
$$\ell(w) - 1 = \#\{\gamma \in \Delta^+ \mid w^{-1} \gamma \in \Delta^-\} - 1$$
■

We are now ready to state the main result.

Theorem 1: Consider any reduced decomposition of the longest elt
$$w_0 = s_{i_1} s_{i_2} \ldots s_{i_\ell} \qquad (\underline{\text{note}}: \ell = |\Delta^+|)$$
Then, the ordered products
$$\{T_{i_1} T_{i_2} \ldots T_{i_{\ell-1}} (E_{i_\ell})^{a_\ell} \cdot \ldots \cdot T_{i_1}(E_{i_2})^{a_2} \cdot E_{i_1}^{a_1} \mid a_1, \ldots, a_\ell \in \mathbb{Z}_{\geq 0}\} \text{ - basis of } U_q^+$$

▶ As above products are lin. indep, it's equivalent to $U^+[w_0] = U_q^+$. However, due to Lemma 2 and $w_0^{-1}(\alpha_i) \in \Delta^-$ $\forall i$, we see that $U^+[w_0]$ is stable under right multiplication by all E_i. As $1 \in U^+[w_0]$, the equality $U^+[w_0] = U_q^+$ follows. ■

<u>Remark</u>: a) Completely analogously, $U^-[w_0] = U_q^- \Rightarrow$ get bases of U_q^-

b) As follows from Fact 2 of Lecture 32, we have
$$\{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \ldots s_{i_{\ell-1}}(\alpha_{i_\ell})\} = \Delta^+$$
while $E_{s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})} := T_{i_1} \ldots T_{i_{k-1}}(E_{i_k}) \in (U_q^+)_{s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})}$
is the "q-analogue" of the root generator of $\mathfrak{g}$.