Lecture #34

Last time: a) For any reduced decomposition of the longest element $w_0 \in W$
$w_0 = s_{i_1} s_{i_2} \ldots s_{i_\ell}$ (here, $\ell = \ell(w_0) = |\Delta^+|$) define the sequence
$\gamma_1 := \alpha_{i_1},\ \gamma_2 := s_{i_1}(\alpha_{i_2}), \ldots, \gamma_\ell = s_{i_1} \ldots s_{i_{\ell-1}}(\alpha_{i_\ell})$. Then, $\Delta^+ = \{\gamma_1, \gamma_2, \ldots, \gamma_\ell\}$

b) For any $\gamma \in \Delta^+$ define the "root generator" $E_\gamma \in U_q^+$ as follows:
if $\gamma = \gamma_k$ with $1 \le k \le \ell$, then $E_\gamma = E_{\gamma_k} := T_{i_1} \ldots T_{i_{k-1}}(E_{i_k})$. This definition is well-defined as shown last time in that $E_{\alpha_j} = E_j \ \forall j \in I$

c) The ordered monomials $\overleftarrow{\prod\limits_{1\le k \le \ell}} E_{\gamma_k}^{a_k} := E_{\gamma_\ell}^{a_\ell} \cdot E_{\gamma_{\ell-1}}^{a_{\ell-1}} \cdot \ldots \cdot E_{\gamma_2}^{a_2} \cdot E_{\gamma_1}^{a_1}$ form a basis of U_q^+. This is usually called the PBW-basis of U_q^+ (as E_{γ_k} clearly q-deforms the root generator $\tilde{s}_{i_1} \ldots \tilde{s}_{i_{k-1}}(e_{i_k}) \in \mathfrak{g}_{\gamma_k}$)

Remarks: a) Completely analogously the products $\overrightarrow{\prod\limits_{1\le k\le \ell}} E_{\gamma_k}^{a_k} := E_{\gamma_1}^{a_1} E_{\gamma_2}^{a_2} \ldots E_{\gamma_\ell}^{a_\ell}$ also form a basis of U_q^+.

b) Analogously, one also obtains the PBW-type bases for U_q^-:
$\{\overleftarrow{\prod\limits_{1\le k\le \ell}} F_{\gamma_k}^{a_k} \mid a_k \ge 0\}$ as well as $\{\overrightarrow{\prod\limits_{1\le k\le \ell}} F_{\gamma_k}^{a_k} \mid a_k \ge 0\}$ are bases of U_q^-,
where $F_{\gamma_k} := T_{i_1} T_{i_2} \ldots T_{i_{k-1}}(F_{i_k})$.

[Exercise: Verify that F_{γ_k} coincides with $\omega(E_{\gamma_k})$ up to a constant in $\pm q^{\mathbb{Z}}$.

c) Evoking the triangular decomposition $U_q^- \otimes U_q^0 \otimes U_q^+ \xrightarrow[\text{mult}]{\text{v.sp}} U_q(\mathfrak{g})$ of Lecture 19 and the obvious basis $\{\prod\limits_{i\in I} K_i^{p_i} \mid p_i \in \mathbb{Z}\}$ of U_q^0, we obtain the bases of the whole quantum group $U_q(\mathfrak{g})$.

Notably, one likewise obtains the PBW-type bases of various "sequential" subalgebras of U_q^+ & U_q^-. We shall focus our attention on U_q^+ for brevity.

Claim (Levendorskii–Soibelman property): For any $1 \le a < b \le \ell$:
$$E_{\gamma_b} \cdot E_{\gamma_a} - q^{-(\gamma_a, \gamma_b)} \cdot E_{\gamma_a} E_{\gamma_b} \in \mathrm{Span}\{\overleftarrow{\prod\limits_{a<c<b}} E_{\gamma_c}^{a_c} \mid a_c \ge 0\} = \mathrm{Span}\{\overrightarrow{\prod\limits_{a<c<b}} E_{\gamma_c}^{a_c}\}$$

This property allows to prove the following important result:

Proposition 1: For any $a \le b$, let $U_{a,b}^+[w_0]$ denote the subalgebra generated by $\{E_{\gamma_c} \mid a \le c \le b\}$. Then both $\{\overleftarrow{\prod\limits_{a\le c\le b}} E_{\gamma_c}^{a_c}\}$ and $\{\overrightarrow{\prod\limits_{a\le c\le b}} E_{\gamma_c}^{a_c}\}$ form bases of $U_{a,b}^+[w_0]$

Let's consider the simplest example of the above claim — corresponding to $\mathfrak{g}=\mathfrak{sl}_3$

Example: $\mathfrak{g}=\mathfrak{sl}_3$, $W=\langle s_1,s_2\rangle\simeq S_3$, $w_0=s_1s_2s_1$ – reduced decomposition $(i_1=1, i_2=2, i_3=1)$

Then: $\gamma_1=\alpha_1$, $\gamma_2=s_1(\alpha_2)=\alpha_1+\alpha_2$, $\gamma_3=s_1s_2(\alpha_1)=\alpha_2$

$$\Downarrow$$

$$E_{\gamma_1}=E_1,\quad E_{\gamma_2}=E_1E_2-q^{-1}E_2E_1,\quad E_{\gamma_3}=E_2.$$

- $\underline{a=1,\ b=2}$ in Claim

$$\underbrace{E_{\gamma_2}\cdot E_{\gamma_1}-q^{-(\alpha_1,\alpha_1+\alpha_2)}E_{\gamma_1}E_{\gamma_2}}=0 \quad (\text{as there are no } 1<c<2!)$$

$$E_1E_2E_1-q^{-1}E_2E_1^2-q^{-1}E_1^2E_2+q^{-2}E_1E_2E_1=-q^{-1}\left(E_1^2E_2-(q+q^{-1})E_1E_2E_1+E_2E_1^2\right)$$

And thus the above equality is nothing but the qSerre rel-n.

- $\underline{a=2,\ b=3}$ in Claim

$$\underbrace{E_{\gamma_3}\cdot E_{\gamma_2}-q^{-(\alpha_2,\alpha_1+\alpha_2)}E_{\gamma_2}E_{\gamma_3}}=0 \quad (\text{as there are no } 2<c<3!)$$

$$E_2E_1E_2-q^{-1}E_2^2E_1-q^{-1}E_1E_2^2+q^{-2}E_2E_1E_2=-q^{-1}\left(E_2^2E_1-(q+q^{-1})E_2E_1E_2+E_1E_2^2\right)$$

And thus the above equality is nothing but the qSerre rel-n.

- $\underline{a=1,\ b=3}$ in Claim

$$\underbrace{E_{\gamma_3}E_{\gamma_1}-q^{-(\alpha_1,\alpha_2)}E_{\gamma_1}E_{\gamma_3}}\in \mathrm{Span}\{E_{\gamma_2}^{a_2}\mid a_2\geq 0\}$$

$$E_2E_1-qE_1E_2=-q\cdot E_{\gamma_2}\quad ,\text{ which confirms above as well!}$$

Rmk: As U_q^+ is graded by $Q_+=\bigoplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$, and $\deg(E_{\gamma_k})=\gamma_k$, in the Claim above one can restrict $\prod_{a<c<b}E_{\gamma_c}^{a_c}$ to those s.t. $\sum a_c\gamma_c=\gamma_a+\gamma_b$.

Rmk: Finally, we note that for the above discussion there is no reason to start from a reduced decomposition of the longest elt w_0. Indeed, given any $w\in W_0$ and its reduced decomposition $w=s_{j_1}\ldots s_{j_p}$ one can concatenate it with a reduced decomposition of $w^{-1}w_0$ on the right or of w_0w^{-1} on the left to get a reduced decomposition of w_0 (which uses $l(w_0w^{-1})=l(w^{-1}w_0)=l(w_0)-l(w)$!!)

In particular, we obtain that $\underbrace{U^+[w] \text{ is a subalgebra of } U_q^+}_{\text{see Remark on page 3 of Lecture 33}}$

The other reason why the above PBW-bases are so useful is that they allow to write down more explicitly Θ and consecutively $R^\vee$ of Lectures 28-29. This is based on the following result:

Theorem 1: Pick a reduced decomposition $w_0 = s_{i_1} s_{i_2} \ldots s_{i_l}$ and define $\{\gamma_k, E_{\gamma_k}, F_{\gamma_k} \mid 1\le k\le l\}$ as above. Then:

$$\left(F_{\gamma_l}^{b_l} \cdot \ldots \cdot F_{\gamma_1}^{b_1},\ E_{\gamma_l}^{a_l}\cdot \ldots \cdot E_{\gamma_1}^{a_1}\right) = \begin{cases} \prod_{k=1}^{l} (-1)^{a_k}\cdot q_{\gamma_k}^{\frac{a_k(a_k-1)}{2}} \cdot \dfrac{[a_k]_{q_{\gamma_k}}!}{(q_{\gamma_k}-q_{\gamma_k}^{-1})^{a_k}} & \text{if } b_k = a_k\ \forall k \\ 0, & \text{otherwise} \end{cases}$$

$(\,,\,)$ - non-degenerate pairing $U_q^{\le} \times U_q^{\ge} \to \Bbbk$ from Lectures 24-25

$q_{\gamma_k} = q^{\frac{1}{2}(\gamma_k,\gamma_k)}$

Recall now our definition of Θ in Lecture 28:

$$\Theta = \sum_{\mu\in Q_+} \sum_{i=1}^{N_\mu = \dim (U_q^+)_\mu} y_i^\mu \otimes x_i^\mu$$ where $\{x_i^\mu\}_{i=1}^{N_\mu}$ and $\{y_i^\mu\}_{i=1}^{N_\mu}$ are dual bases of U_q^+ and U_q^-

Combining with the above result, we see that if $q \ne \sqrt[?]{1}$, then picking $\{\overleftarrow{\prod} E_{\gamma_k}^{a_k}\}$ as a basis for U_q^+, the dual basis is $\overleftarrow{\prod\limits_{1\le k\le l}} \left\{F_{\gamma_k}^{a_k} \cdot (-1)^{a_k}\cdot q_{\gamma_k}^{-\frac{a_k(a_k-1)}{2}} \cdot \dfrac{(q_{\gamma_k}-q_{\gamma_k}^{-1})^{a_k}}{[a_k]_{q_{\gamma_k}}!}\right\}$

Therefore:

$$\boxed{\Theta = \sum_{a_1,\ldots,a_l\ge 0} F_{\gamma_l}^{a_l}\cdot\ldots\cdot F_{\gamma_1}^{a_1} \otimes E_{\gamma_l}^{a_l}\cdot\ldots\cdot E_{\gamma_1}^{a_1} \cdot \prod_{k=1}^{l} (-1)^{a_k} q_{\gamma_k}^{-\frac{a_k(a_k-1)}{2}} \frac{(q_{\gamma_k}-q_{\gamma_k}^{-1})^{a_k}}{[a_k]_{q_{\gamma_k}}!}}$$

Remark: The explicit constant in the end was already encountered in [Lecture 24, Lemma 6]: $(F_i^n, E_i^n) = (-1)^n q_i^{\frac{n(n-1)}{2}} \cdot \dfrac{[n]_{q_i}!}{(q_i - q_i^{-1})^n}$.

Finally, we also note that the above formula for Θ can be factorized:

$$\boxed{\Theta = \Theta^{[l]}\,\Theta^{[l-1]} \cdots \Theta^{[1]} \quad \text{with} \quad \Theta^{[k]} = \sum_{r\ge 0} (-1)^r q_{\gamma_k}^{-\frac{r(r-1)}{2}} \frac{(q_{\gamma_k}-q_{\gamma_k}^{-1})^r}{[r]_{q_{\gamma_k}}!} F_{\gamma_k}^r \otimes E_{\gamma_k}^r}$$

where each $\Theta^{[k]}$ looks precisely as our Θ for $\mathfrak{g} = \mathfrak{sl}_2$ back from Lect 13-14. Note that it can be standardly written in terms of the q-exponent of Lect 7:

$$\boxed{\Theta^{[k]} = e_{q_{\gamma_k}^2}\left((q_{\gamma_k}^{-1} - q_{\gamma_k}) F_{\gamma_k} \otimes E_{\gamma_k}\right)}$$

Upshot: The overall Θ factorized into the product of q-exponents of $*\,F_\gamma \otimes E_\gamma$ as γ ranges over all Δ^+ (the latter being $\mathfrak{sl}_2$-computational

Lecture #34

Finally, with Theorem 1 at hand, we can actually replace all "tQ-assumption" by "q not a root of 1" ($q \neq \sqrt[?]{1}$) in our earlier lectures. In particular:

Proposition 2: The q-Harish-Chandra isomorphism $HC: Z_q(\mathfrak{g}) \xrightarrow{\sim} (U^0_{ev})^W$ holds whenever q is not a root of 1.

Sketch of the proof

▸ According to PBW theorem from last time and Theorem 1 above, we see that if $q \neq \sqrt[?]{1}$, then the pairing $(\ ,\ ): U_q^{\leq} \times U_q^{\geq} \to \mathbb{k}$ is nondegenerate. Hence, also the pairing $\langle\ ,\ \rangle: U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \to \mathbb{k}$ is non-degenerate, see [Lecture 26, Proposition 2]

Thus, arguing as in Lecture 27, we immediately obtain $(U^0_{ev})^W \subseteq HC(Z_q(\mathfrak{g}))$ (indeed, using quantum traces of $\{L(\lambda) \mid \lambda \in P_+ \cap \frac{1}{2}Q\}$ and pairing $\langle\ ,\ \rangle$ above, we proved that $HC(Z_q(\mathfrak{g}))$ contains all $\{Av(\nu) \mid \nu \in 2P \cap Q\}$)

In particular, if $z_\nu \in Z_q(\mathfrak{g})$ is such that $HC(z_\nu) = Av(\nu)$, then we have:

$$\boxed{\pi(z_\nu) = \Big(\sum_{w \in W} q^{(w\nu, \rho)} K_{w\nu}\Big) \cdot \frac{1}{|Stab_W(\nu)|}}$$

Lemma 1: If $\lambda_1 \in P_+$, $\lambda_2 \in P$ are such that $Z_q(\mathfrak{g})$ acts on the Verma modules $M(\lambda_1)$ and $M(\lambda_2)$ by the same characters, then $\lambda_2 + \rho \in W(\lambda_1 + \rho)$

▸ It suffices to compare action of $\{z_\nu\}_{\nu \in 2P \cap Q}$ on these Verma modules. Indeed, we get: $\sum_{w \in W} q^{(w\nu, \rho)} \cdot q^{(w\nu, \lambda_1)} = \sum_{w \in W} q^{(w\nu, \rho)(w\nu, \lambda_2)}$ which can be equivalently written as $\sum_{w \in W} q^{(\nu, w(\lambda_1 + \rho))} = \sum_{w \in W} q^{(\nu, w(\lambda_2 + \rho))}$

But $\lambda_1 \in P_+ \Rightarrow \lambda_1 + \rho$ is strictly dominant $\Rightarrow$ all $\{w(\lambda_1 + \rho) \mid w \in W\}$ are pairwise distinct

Hence, evoking Artin's lemma on characters, we get $\lambda_2 + \rho \in W(\lambda_1 + \rho)$. ☒

. This lemma allows us to easily conclude that $\tilde{L}(\lambda) \twoheadrightarrow L(\lambda)$ is actually an isomorphism, compare to [Lecture 22, Theorem 1]. Indeed, take any Jordan-Hölder filtration of $\tilde{L}(\lambda)$, then since the center acts on all subsequent quotients in the same way, we get:

(Continuation)

$\mu+\rho \in W(\lambda+\rho)$ for any subsequent quotient $\simeq L(\mu)$.

However, the W-orbit $W(\lambda+\rho)$ contains only one strictly dominant weight $\Rightarrow \mu=\lambda$. But $\dim(\tilde{L}(\lambda)_\lambda)=1=\dim(L(\lambda)_\lambda)$. Thus, $\tilde{L}(\lambda) \xrightarrow{\sim} L(\lambda)$ as claimed.

- With this result in hands, one obtains [Lecture 22, Thm 2] for $q \neq \sqrt[?]{1}$.
- Combining this semisimplicity with [Homework 4, Problem 4], we finally get:

Proposition 3: If $q\neq \sqrt[?]{1}$ and $u \in U_q(\mathfrak{g})$ acts by zero on all $\{L(\lambda) \mid \lambda \in P_+\}$, then $u=0$

Finally, having established Prop 3 above, we can now apply our argument from [Lecture 23, Lemma 2] to show that π, hence also HC, is injective.

This proves that HC - isomorphism whenever $q \neq \sqrt[?]{1}$