Lecture #35

Goal: Introduce the general concept of tensor categories

We shall start by recalling some basics about categories

Def: A category $\mathcal{C}$ consists of

- a class $Ob(\mathcal{C})$ of objects of the category
- a class $Hom(\mathcal{C})$ of morphisms of the category
- sourse class function $Hom(\mathcal{C}) \xrightarrow{s} Ob(\mathcal{C})$
- target class function $Hom(\mathcal{C}) \xrightarrow{t} Ob(\mathcal{C})$ } usually this is perceived by saying that $\forall X, Y \in Ob(\mathcal{C})$ one has $Hom_{\mathcal{C}}(X,Y) = \{f \in Hom(\mathcal{C}) \mid {s(f)=X \atop t(f)=Y}\}$
- identity class function $Ob(\mathcal{C}) \xrightarrow{id} Hom(\mathcal{C})$ } this is the identity morphism $\forall X \in Ob(\mathcal{C})$ have $id_X \in Hom_{\mathcal{C}}(X,X)$
- composition of morphisms $\underbrace{Hom(\mathcal{C}) \underset{Ob(\mathcal{C})}{\times} Hom(\mathcal{C})}_{\{(g,f) \in Hom(\mathcal{C}) \times Hom(\mathcal{C}) \mid t(f) = s(g) \in Ob(\mathcal{C})\}} \xrightarrow{g \circ f = gf} Hom(\mathcal{C})$

subject to the following compatibilities

1) $id_Y \circ f = f \circ id_X = f \quad \forall X, Y \in Ob(\mathcal{C}),\ f \in Hom_{\mathcal{C}}(X,Y)$

2) $(h \circ g) \circ f = h \circ (g \circ f) \quad \forall X, Y, Z, W \in Ob(\mathcal{C}),\ f \in Hom_{\mathcal{C}}(X,Y),\ g \in Hom_{\mathcal{C}}(Y,Z),\ h \in Hom_{\mathcal{C}}(Z,W)$

When $f \in Hom_{\mathcal{C}}(X,Y)$ we shall often write $f: X \to Y$

Def: $f: X \to Y$ is an isomorphism if $\exists g: Y \to X$ s.t. $g \circ f = id_X$ & $f \circ g = id_Y$

Examples: 1) Category Set : objects of Set are all sets
morphisms of Set are all functions between two sets

2) Category $Vect_{\mathbb{k}}$: objects are vector spaces over $\mathbb{k}$
morphisms are linear maps b/w two vector spaces

2') $Vect_{\mathbb{k}}^{f.d.}$: as above but only fin. dimensional v. spaces

3) For any algebra A, have the category A-Mod : objects are left A-modules
morphisms are A-mod homom.

4) Category Grp : objects are all groups
morphisms are all gp homom-s

5) Category Alg : objects are all algebras
morphisms are algebra homom-s

6) Given any set I with a binary operation $\leq$ s.t. (i) $x \leq x \ \forall x \in \mathbb{Z}$ and (ii) $x \leq y,\ y \leq z \Rightarrow x \leq z$, one has a category $\mathcal{C}$ with $Ob(\mathcal{C}) = I$ and $Hom(\mathcal{C}) = \{(x,y) \in I \times I \mid x \leq y\}$, i.e. $Hom_{\mathcal{C}}(x,y) = \begin{cases} 1 \text{ elt class if } x \leq y \\ \emptyset \text{ if } x \not\leq y. \end{cases}$

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A few relevant general definitions are:

- a <u>subcategory</u> $\mathcal{D}$ of the category $\mathcal{C}$ consists of subclasses $Ob(\mathcal{D}) \subset Ob(\mathcal{C})$, $Hom(\mathcal{D}) \subset Hom(\mathcal{C})$ stable under id, s, t, and composition maps in $\mathcal{C}$.
- a <u>groupoid</u> is a category $\mathcal{C}$ in which any morphism is an isomorphism
- a <u>product</u> of categories $\mathcal{C}$ and $\mathcal{D}$ is a category denoted $\mathcal{C} \times \mathcal{D}$ with
  $Ob(\mathcal{C}\times\mathcal{D}) = Ob\,\mathcal{C} \times Ob\,\mathcal{D} = \{(X,Y) \mid X \in Ob(\mathcal{C}),\ Y \in Ob(\mathcal{D})\}$
  $Mor(\mathcal{C}\times\mathcal{D}) = Mor(\mathcal{C}) \times Mor(\mathcal{D})$, i.e. $Hom_{\mathcal{C}\times\mathcal{D}}((X,Y),(X',Y')) = \left\{(f,g) \,\middle|\, \begin{matrix} f: X \to X' \\ g: Y \to Y' \end{matrix}\right\}$

<u>Example</u> (of groupoids): a) For any group G, consider a category $\mathcal{C}$ with only 1 object, denoted x, and $Hom_{\mathcal{C}}(x,x) = G$ with the composition being the group product. Obviously, $\mathcal{C}$-groupoid

b) More generally, given a family of groups $(G_i)_{i\in I}$, indexed by some set I, consider the category $\mathcal{G}$ with $Ob(\mathcal{G}) = I$ and $Hom_{\mathcal{G}}(i,j) = \begin{cases} G_i & \text{if } j=i \\ \emptyset & \text{if } j\neq i, \end{cases}$ with composition being the group product in each G_i. Obviously, $\mathcal{G}$-groupoid.

The next key notion is that of functors:

<u>Def</u>: Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ (denoted $\mathcal{C} \to \mathcal{D}$) is

- a map $F: Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$
- a map $F: Hom(\mathcal{C}) \longrightarrow Hom(\mathcal{D})$

satisfying the following compatibilities:

1) $F(id_X) = id_{F(X)}$
2) if $f: X \to Y$ then $F(f): F(X) \to F(Y)$ (i.e. $s(F(f)) = F(s(f))$, $t(F(f)) = F(t(f))$)
3) if $\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$ then $\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g\circ f)"'] & F(Y) \arrow[r, "F(g)"] & F(Z) \end{tikzcd}$ i.e. $F(g\circ f) = F(g)\circ F(f)$ whenever $g\circ f$ is defined.

Note that the composition of functors is again a functor, and for $\mathcal{C} = \mathcal{D}$ we always have the identity functor $id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$. This results in:

<u>Example</u>: The class of all categories with morphisms from $\mathcal{C}$ to $\mathcal{D}$ being all the functors $\mathcal{C} \to \mathcal{D}$ give rise to a category Cat.

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Another crucial notion is that of natural transformations

Def: Given two categories $\mathcal{C}, \mathcal{D}$ and two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* η from F to G (again denoted by $\eta : F \to G$) is a family of morphisms (called components of η) $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$ (parametrized by $X \in Ob(\mathcal{C})$) such that $\forall f : X \to Y$ in $\mathcal{C}$ the following commutes

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

Note that a composition of natural transformations is again a natural transformation, and for any functor $F : \mathcal{C} \to \mathcal{D}$ we have the identity natural transformation $id_F : F \to F$ s.t. $\eta_X = id_{F(X)}$ $\forall X \in Ob(\mathcal{C})$. This results in:

Example: Given two categories $\mathcal{C}, \mathcal{D}$, there is a category $\mathcal{D}^{\mathcal{C}}$ with $Ob(\mathcal{D}^{\mathcal{C}})$ = functors $\mathcal{C} \to \mathcal{D}$, $Hom(\mathcal{D}^{\mathcal{C}})$ = natural transformations

Def: A natural transformation $\eta : F \to G$ is a natural isomorphism if $\forall X \in Ob(\mathcal{C})$ $\eta_X : F(X) \to G(X)$ is an isomorphism in category $\mathcal{D}$.

With the above concept in mind, one can now define equivalence of categories

Def: Given two categories $\mathcal{C}, \mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if $\exists$ functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms

$$\eta : id_{\mathcal{D}} \to FG \quad , \quad \Theta : GF \to id_{\mathcal{C}}$$

In fact, one has a simple criteria for $F : \mathcal{C} \to \mathcal{D}$ to be an equivalence purely in terms of F (rather than existence of some G). To state the result we introduce some new terminology:

- functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if $\forall Y \in Ob(\mathcal{D})$ $\exists X \in Ob(\mathcal{C})$ s.t. $F(X) \simeq Y$ in $\mathcal{D}$
- functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if $\forall X, Y \in Ob(\mathcal{C})$, the map $F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$ is injective
- functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if the above is bijective

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With these notions, we have the following simple result:

Proposition 1: Given two categories $\mathcal{C}, \mathcal{D}$ and a functor $F: \mathcal{C} \to \mathcal{D}$, we have:
F - equivalence of categories $\iff$ F is essentially surjective & fully faithful

Rmk: As an important application, given a subcollection $Ob(\mathcal{C}')$ of objects $Ob(\mathcal{C})$ s.t. $\forall X \in Ob(\mathcal{C})$ $\exists Y \in Ob(\mathcal{C}')$ isomorphic to X, the full subcategory $\mathcal{C}'$ of $\mathcal{C}$ whose objects are elts of $Ob(\mathcal{C}')$ and $Hom_{\mathcal{C}'}(X', Y') := Hom_{\mathcal{C}}(X', Y')$ is equivalent to $\mathcal{C}$ under the inclusion $F: \mathcal{C}' \to \mathcal{C}$.

$\boxed{\Rightarrow}$

- Evoking $\eta: id_{\mathcal{D}} \to FG$ - natural isomorphism, we get $Y \xrightarrow{\sim} F(G(Y)) \Rightarrow F$ - essent. surj.
- Evoking $\Theta: GF \to id_{\mathcal{C}}$, we obtain $\forall f: X_1 \to X_2$ in $\mathcal{C}$ the following commutative diagram:

$$\begin{tikzcd}
GF(X_1) \arrow[r, "\Theta_{X_1}", "\sim"'] \arrow[d, "GF(f)"'] & X_1 \arrow[d, "f"] \\
GF(X_2) \arrow[r, "\Theta_{X_2}", "\sim"'] & X_2
\end{tikzcd}$$

If $F(f_1) = F(f_2) \Rightarrow GF(f_1) = GF(f_2) \Rightarrow f_1 = f_2$
B/c $\Theta_{X_1}, \Theta_{X_2}$ - isom.

Hence: F - faithful. Likewise: G - faithful

- Finally, let's verify that F is fully faithful. To this end, pick $g: F(X_1) \to F(X_2)$ and define $f: X_1 \to X_2$ via above diagram i.e. set $f := \Theta_{X_2} \circ G(g) \circ \Theta_{X_1}^{-1}$
Clearly $GF(f) = G(g)$. But G is faithful, hence, $g = F(f)$.

$\boxed{\Leftarrow}$.

Assuming F is essentially surjective and fully faithful, we shall now construct the functor G as in definition of equivalence.

- For any $Y \in Ob(\mathcal{D})$, choose $G(Y) \in Ob(\mathcal{C})$ and an isomorphism $\eta_Y: Y \xrightarrow{\sim} F(G(Y))$
- For any $Y_1 \xrightarrow{g} Y_2$, we define $G(g): G(Y_1) \to G(Y_2)$ by specifying
$$F(G(g)): FG(Y_1) \xrightarrow{\eta_{Y_1}^{-1}} Y_1 \xrightarrow{g} Y_2 \xrightarrow{\eta_{Y_2}} F(G(Y_2))$$
which determines $G(g)$ uniquely as F is fully faithful.

By construction of G, we also have a natural isomorphism $\eta: id_{\mathcal{D}} \to FG$.
Finally, $\Theta_V: GF(V) \to V$ is uniquely determined (due to fully faithfulness of F) by $F(\Theta_V): FGF(V) \to F(V)$ which we set to be $\eta_{F(V)}^{-1}$. Verify it works!

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The last key concept on categories for today is the notion of adjoint functors

Def: Given two categories $\mathcal{C}, \mathcal{D}$ and two functors $\mathcal{D} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{C}$ one says "G is left adjoint to F" or equivalently "F is right adjoint to G" if there are natural transformations

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "\mathrm{id}_{\mathcal{D}}", ""{name=U, below}] \arrow[r, bend right=50, "FG"', ""{name=L}] & \mathcal{D} \arrow[Rightarrow, from=U, to=L, "\eta"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF", ""{name=U, below}] \arrow[r, bend right=50, "\mathrm{id}_{\mathcal{C}}"', ""{name=L}] & \mathcal{C} \arrow[Rightarrow, from=U, to=L, "\Theta"] \end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{i.e. } \eta : \mathrm{id}_{\mathcal{D}} \to FG} \qquad\qquad \underbrace{\qquad\qquad\qquad}_{\text{i.e. } \Theta : GF \to \mathrm{id}_{\mathcal{C}}}$$

such that

$$\begin{tikzcd} F(V) \arrow[r, "\eta_{F(V)}"] \arrow[rr, bend right, "\mathrm{id}_{F(V)}"'] & FGF(V) \arrow[r, "F(\Theta_V)"] & F(V) \end{tikzcd}, \qquad \begin{tikzcd} G(W) \arrow[r, "G(\eta_W)"] \arrow[rr, bend right, "\mathrm{id}_{G(W)}"'] & GFG(W) \arrow[r, "\Theta_{G(W)}"] & G(W) \end{tikzcd}$$

commute

Much more often one uses the following equivalent criteria for adjointness:

Proposition 2: $\mathcal{D} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{C}$ are adjoint $\iff$ $\forall\, Y \in \mathrm{Ob}(\mathcal{D}),\ X \in \mathrm{Ob}(\mathcal{C})$ there is a natural bijection $\varphi_{XY} : \mathrm{Hom}_{\mathcal{C}}(G(Y), X) \xrightarrow{\text{1-to-1}} \mathrm{Hom}_{\mathcal{D}}(Y, F(X))$

here, natural means that we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(G(Y'), X) \arrow[r, "\varphi_{XY'}"'] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(Y', F(X)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{C}}(G(Y), X) \arrow[r, "\varphi_{XY}"'] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(Y, F(X)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{C}}(G(Y), X') \arrow[r, "\varphi_{X'Y}"'] & \mathrm{Hom}_{\mathcal{D}}(Y, F(X')) \end{tikzcd}$$

$\forall\ f : X \to X'$
$g : Y \to Y'$
where vertical maps arise through taking Hom from/to $G(Y) \to G(Y')$, $F(X) \to F(X')$

Exercise: Prove this proposition in full details (if you never did it before).

Hint: $\boxed{\Rightarrow}$ define $\varphi_{XY}(\underset{\in \mathrm{Hom}_{\mathcal{C}}(G(Y),X)}{f}) := F(f) \circ \eta_Y : Y \xrightarrow{\eta_Y} FG(Y) \xrightarrow{F(f)} F(X)$

and show that its inverse is $\forall g \in Y \to F(X) \rightsquigarrow G(Y) \xrightarrow{G(g)} GF(X) \xrightarrow{\Theta_X} X$

$\boxed{\Leftarrow}$ define $\eta_Y := \varphi_{G(Y),Y}(\mathrm{id}_{G(Y)})$, $\Theta_X := \varphi^{-1}_{X,F(X)}(\mathrm{id}_{F(X)})$

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Let's conclude with examples of adjoint functors that we already encountered:

Examples (of adjoint functors):

1) X - set $\rightsquigarrow \mathbb{k}\langle X\rangle$ - free algebra in X

This defines a functor Set $\longrightarrow$ Alg which is left adjoint to the forgetful functor Alg $\longrightarrow$ Set (forgetful means we forget algebra structure and just view els of an algebra as a set)

2) $\mathfrak{g}$ - Lie algebra $\rightsquigarrow U(\mathfrak{g})$ - universal enveloping of $\mathfrak{g}$.

This defines a functor $\underbrace{\text{Lie}}_{\text{category of Lie algebras}} \longrightarrow \text{Alg}$ which is left adjoint to the functor Alg $\longrightarrow$ Lie that given algebra A views it as a Lie algebra with $[a,b] := a \cdot b - b \cdot a$

3) R - integral domains $\rightsquigarrow$ Frac(R) - field of fractions defines a functor

$\{\text{integral domains}\} \longrightarrow \{\text{fields}\}$

left adjoint to the forgetful functor.