# Lecture #36

Today: Tensor categories

We start from the following data:

- $\mathcal{C}$ - category
- $\otimes$ - functor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$, called a tensor product

As recalled last time, the above $\otimes$ thus assigns

$\forall X, Y\in Ob(\mathcal{C}) \rightsquigarrow X\otimes Y\in Ob(\mathcal{C})$

$\forall f: X\to X', g: Y\to Y' \rightsquigarrow f\otimes g: X\otimes Y\to X'\otimes Y'$.

so that $id_X\otimes id_Y = id_{X\otimes Y}$ and $(f'\otimes g')\circ(f\otimes g) = (f'\circ f)\otimes(g'\circ g)$ $\forall f': X'\to X'', g'': Y'\to Y''$.

Basic example: $\mathcal{C} = Vect_{\mathbb{k}}$, $\otimes: \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is the usual tensor product over $\mathbb{k}$

Motivated by this example and the obvious isomorphisms

$$(U\otimes V)\otimes W \simeq U\otimes(V\otimes W), \quad \mathbb{k}\otimes V\simeq V, \quad V\otimes\mathbb{k}\simeq V$$

we shall now introduce some additional structures that mimick these isomorphisms

- **Def**: An associativity constraint for $\otimes$ is a natural isomorphism
$a: \otimes(\otimes\times id)\to\otimes(id\times\otimes)$ also written as $\begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, bend left, "\otimes(\otimes\times id)"] \arrow[r, bend right, "\otimes(id\times\otimes)"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{C} \end{tikzcd}$

This structure amounts for a family of isomorphisms $\{a_{X,Y,Z} \mid X,Y,Z\in Ob(\mathcal{C})\}$

$$a_{X,Y,Z}: (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z) \quad \text{- isomorphism of objects}$$

natural/functorial in the following sense:

$\underbrace{\forall f: X\to X', g: Y\to Y', h: Z\to Z'}_{\text{morphisms in } \mathcal{C}}$ $\xrightarrow[\text{diagram}]{\text{have commutative}}$
$$\begin{tikzcd} (X\otimes Y)\otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "(f\otimes g)\otimes h"'] & X\otimes(Y\otimes Z) \arrow[d, "f\otimes(g\otimes h)"] \\ (X'\otimes Y')\otimes Z' \arrow[r, "a_{X',Y',Z'}"] & X'\otimes(Y'\otimes Z') \end{tikzcd}$$

- Furthermore, one imposes the following Pentagon axiom on associativity constraint

$$\begin{tikzcd} ((X\otimes Y)\otimes Z)\otimes W \arrow[r, "a_{X,Y,Z}\otimes id_W"] \arrow[d, "a_{X\otimes Y,Z,W}"'] & (X\otimes(Y\otimes Z))\otimes W \arrow[r, "a_{X,Y\otimes Z,W}"] & X\otimes((Y\otimes Z)\otimes W) \arrow[d, "id_X\otimes a_{Y,Z,W}"] \\ (X\otimes Y)\otimes(Z\otimes W) \arrow[rr, "a_{X,Y,Z\otimes W}"] & & X\otimes(Y\otimes(Z\otimes W)) \end{tikzcd}$$
– commutes $\underbrace{\forall X,Y,Z,W}_{\in Ob(\mathcal{C})}$

- To define an analogue of $\mathbb{k}\otimes V\simeq V\simeq V\otimes\mathbb{k}$ in $Vect_{\mathbb{k}}$, we have to add the following data: an object $I\in Ob(\mathcal{C})$, called the unit, satisfying left & right unit constraints:

**Def**: A left unit constraint w.r.t. $I$ is a natural isomorphism $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\otimes(I\times id_{\mathcal{C}})"] \arrow[r, bend right, "id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow l"] & \mathcal{C} \end{tikzcd}$

A right unit constraint w.r.t. $I$ is a natural isom. $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\otimes(id_{\mathcal{C}}\times I)"] \arrow[r, bend right, "id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow r"] & \mathcal{C} \end{tikzcd}$

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The left & right unit constraints w.r.t. $I$ amount to the families of isomorphisms

$$l_X : I\otimes X \longrightarrow X \quad\text{and}\quad r_X : X\otimes I \longrightarrow X \qquad \forall X\in Ob(\mathcal{C})$$

natural in the sense that for any morphism $f: X\to Y$ we have commut. diagr:

$$\begin{array}{ccc} I\otimes X & \xrightarrow{l_X} & X \\ {\scriptstyle id_I\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ I\otimes Y & \xrightarrow{l_Y} & Y \end{array} \quad\text{and}\quad \begin{array}{ccc} X\otimes I & \xrightarrow{r_X} & X \\ \downarrow{\scriptstyle f\otimes id_I} & & \downarrow{\scriptstyle f} \\ Y\otimes I & \xrightarrow{r_Y} & Y \end{array}$$

Moreover, one imposes the following "natural" axiom on $a, l, r$:

Triangle axiom:

$$\begin{array}{ccc} (X\otimes I)\otimes Y & \xrightarrow{a_{X,I,Y}} & X\otimes(I\otimes Y) \\ & {\scriptstyle r_X\otimes id_Y}\searrow \quad \swarrow{\scriptstyle id_X\otimes l_Y} & \\ & X\otimes Y & \end{array} \qquad \forall X,Y\in Ob(\mathcal{C})$$

**Key Definition**: A tensor category $(\mathcal{C},\otimes,I,a,l,r)$ is a category $\mathcal{C}$, equipped with a tensor product functor $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$, an object $I\in Ob(\mathcal{C})$ called the unit, with associativity constraint $a$, left unit constraint $l$, and right unit constraint $r$, such that Pentagon & Triangle axioms hold.

- Back to the basic example of $\mathcal{C}=Vect_{\mathbb{k}}$, $\otimes$ being the usual tensor product, $I=\mathbb{k}$:

  $a: (U\otimes V)\otimes W \longrightarrow U\otimes(V\otimes W)$ maps $(u\otimes v)\otimes w \mapsto u\otimes(v\otimes w)$

  $l: \mathbb{k}\otimes V \longrightarrow V$ maps $1\otimes v\mapsto v$

  $r: V\otimes\mathbb{k}\longrightarrow V$ maps $v\otimes 1\mapsto v$

- As a slight upgrade of $Vect_{\mathbb{k}}$, let's consider the category $A$-Mod of left modules over an algebra $A$. Assume we also have algebra homom-s

$$\Delta: A\longrightarrow A\otimes A \quad\text{and}\quad \varepsilon: A\longrightarrow \mathbb{k} \text{ (ground field)}$$

This allows to define $\otimes: A\text{-Mod}\times A\text{-Mod}\longrightarrow A\text{-Mod}$ as well as equip $\mathbb{k}$ with an $A$-module structure, hence the unit elt $\mathbb{k}\in A$-Mod.

**Lemma 1**: The above data $(A,\Delta,\varepsilon)$ is a bialgebra iff $A$-Mod is a tensor category

Exercise (easy): prove lemma

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Let us now prove some basic properties of the unit object $I$.

**Lemma 2**: For any $X, Y \in Ob(\mathcal{C})$ the following triangles commute:

$$\begin{tikzcd}
(I\otimes X)\otimes Y \arrow[rr, "a_{I,X,Y}"] \arrow[dr, "l_X\otimes id_Y"'] & & I\otimes(X\otimes Y) \arrow[dl, "l_{X\otimes Y}"] \\
& X\otimes Y &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(X\otimes Y)\otimes I \arrow[rr, "a_{X,Y,I}"] \arrow[dr, "r_{X\otimes Y}"'] & & X\otimes(Y\otimes I) \arrow[dl, "id_X\otimes r_Y"] \\
& X\otimes Y &
\end{tikzcd}$$

This result is proved by constructing a big diagram (see p. 283 in Kassel's book):

$\forall A \in Ob(\mathcal{C})$

$$\begin{tikzcd}
((A\otimes I)\otimes X)\otimes Y \arrow[rr] \arrow[dr] \arrow[ddd] & & (A\otimes(I\otimes X))\otimes Y \arrow[dl] \arrow[ddd] \\
& (A\otimes X)\otimes Y \arrow[d] & \\
& A\otimes(X\otimes Y) & \\
(A\otimes I)\otimes(X\otimes Y) \arrow[ur] \arrow[dr] & & A\otimes((I\otimes X)\otimes Y) \arrow[ul] \arrow[dl] \\
& A\otimes(I\otimes(X\otimes Y)) \arrow[uu] &
\end{tikzcd}$$

we skipped labels of arrows
exercise: find them

and observing the following commutative subdiagrams:

1) the outside pentagon – due to the Pentagon axiom
2) the left rectangle – due to naturality of $a$ w.r.t. $\begin{cases} r_A : A\otimes I \to A \\ id_X : X \to X,\ id_Y : Y \to Y. \end{cases}$
3) the right rectangle – due to naturality of $a$ w.r.t. $\begin{cases} l_X : I\otimes X \to X \\ id_A,\ id_Y \end{cases}$
4) the top triangle – due to the Triangle axiom (tensored with $id_Y$)
5) the bottom-left triangle – due to the Triangle axiom.

We claim that 1)-5) together with all arrows being isomorphism imply that the bottom-right triangle commutes. To see it – compose both $\nwarrow id_A\otimes l_X\otimes id_Y$ and $\uparrow id_A\otimes l_{X\otimes Y}$, $\swarrow id_A\otimes a_{I,X,Y}$ with top $\xrightarrow{a_{A,I,X}\otimes id_Y}$ and vertical right $\downarrow a_{A,I\otimes X,Y}$.

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(Continuation)

Thus, we get:

$$id_A \otimes (l_X \otimes id_Y) = (id_A \otimes l_{X\otimes Y}) \circ (id_A \otimes a_{I,X,Y}) \; : \; A\otimes((I\otimes X)\otimes Y) \longrightarrow A\otimes(X\otimes Y)$$

which holds for any $A \in Ob(\mathcal{C})$. In particular, we can take $A = I$ - unit object which immediately implies the commutativity of the first triangle in Lemma

[ **Exercise**: prove the other commutativity ■

**Lemma 3**: If $I$ - unit object, then $l_{I\otimes X} = id_I \otimes l_X$, $r_{X\otimes I} = r_X \otimes id_I$ $\forall X \in Ob(\mathcal{C})$.

By naturality of $l$, we have the following commutative square:

$$\begin{tikzcd} I\otimes(I\otimes X) \arrow[r, "l_{I\otimes X}"] \arrow[d, "id_I\otimes l_X"'] & I\otimes X \arrow[d, "l_X"] \\ I\otimes X \arrow[r, "l_X"] & X \end{tikzcd} \quad \xRightarrow{l_X\text{-isom.}} \quad l_{I\otimes X} = id_I \otimes l_X$$ proving the $1^{st}$ equality

The $2^{nd}$ equality is deduced likewise from

$$\begin{tikzcd} (X\otimes I)\otimes I \arrow[r, "r_{X\otimes I}"] \arrow[d, "r_X\otimes id_I"'] & X\otimes I \arrow[d, "r_X"] \\ X\otimes I \arrow[r, "r_X"] & X \end{tikzcd}$$ ■

**Lemma 4**: If $I$ - unit object, then $r_I = l_I$

Let $a_I = a_{I,I,I} : (I\otimes I)\otimes I \longrightarrow I\otimes(I\otimes I)$. Then:

$$l_I \otimes id_I \overset{\text{Lemma 2}}{=\!=} l_{I\otimes I}\circ a_I \overset{\text{Lemma 3}}{=\!=} (id_I\otimes l_I)\circ a_I \overset{\text{Triangle axiom}}{=\!=} r_I \otimes id_I$$

which implies the claimed $l_I = r_I$ (can ignore $\otimes id_I$, since $r$ - isom.) ■

**Lemma 5**: The endomorphisms $End(I)$ is a commutative monoid for the composition. Moreover, the tensor product of morphisms coincides with composition once we identify $I\otimes I \xrightarrow[\sim]{l_I = r_I} I$, i.e. $f\otimes g = r_I^{-1}\circ(f\circ g)\circ r_I$

By commutativity of

$$\begin{tikzcd} I\otimes I \arrow[r, "r_I"] \arrow[d, "f\otimes id_I"'] & I \arrow[d, "f"] \\ I\otimes I \arrow[r, "r_I"] & I \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I\otimes I \arrow[r, "l_I"] \arrow[d, "id_I\otimes g"'] & I \arrow[d, "g"] \\ I\otimes I \arrow[r, "l_I"] & I \end{tikzcd}$$

, have $f\otimes id_I = r_I^{-1}\circ f\circ r_I$, $id_I\otimes g = l_I^{-1}\circ g\circ l_I$

As $l_I = r_I$ by Lemma 4, and $f\otimes g = (f\otimes id)\circ(id\otimes g) = (id\otimes g)\circ(f\otimes id)$, we derive the above f-la as well as $f\circ g = g\circ f$ ■

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We conclude by introducing general notions of tensor functors, natural tensor transf.

**Def:** Given two tensor categories $(\mathcal{C}, \otimes, I, a, l, r)$ and $(\mathcal{C}', \otimes', I', a', l', r')$, a tensor functor from $\mathcal{C}$ to $\mathcal{C}'$ is a triple $(F, \varphi_0, \varphi_2)$, where

- $F: \mathcal{C} \longrightarrow \mathcal{C}'$ – functor
- $\varphi_0 : I' \longrightarrow F(I)$ – isomorphism in $\mathcal{C}'$
- $\varphi_2 = \{\varphi_2(X,Y) \mid X, Y \in Ob(\mathcal{C})\}$ is a family of isomorphisms in $\mathcal{C}'$

$$\varphi_2(X,Y) : F(X) \otimes' F(Y) \longrightarrow F(X \otimes Y)$$

natural in the sense that all the following diagrams commute:

$$\begin{array}{ccc} I' \otimes' F(X) & \xrightarrow{l'_{F(X)}} & F(X) \\ \downarrow{\scriptstyle \varphi_0 \otimes' id_{F(X)}} & & \uparrow{\scriptstyle F(l_X)} \\ F(I) \otimes' F(X) & \xrightarrow{\varphi_2(I,X)} & F(I \otimes X) \end{array} \qquad \begin{array}{ccc} F(X) \otimes' I' & \xrightarrow{r'_{F(X)}} & F(X) \\ \downarrow{\scriptstyle id_{F(X)} \otimes' \varphi_0} & & \uparrow{\scriptstyle F(r_X)} \\ F(X) \otimes' F(I) & \xrightarrow{\varphi_2(X,I)} & F(X \otimes I) \end{array}$$

$$\begin{array}{ccc} (F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\ \downarrow{\scriptstyle \varphi_2(X,Y) \otimes' id_{F(Z)}} & & \downarrow{\scriptstyle id_{F(X)} \otimes' \varphi_2(Y,Z)} \\ F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\ \downarrow{\scriptstyle \varphi_2(X \otimes Y, Z)} & & \downarrow{\scriptstyle \varphi_2(X, Y \otimes Z)} \\ F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z)) \end{array}$$

**Def:** A natural tensor transformation $\eta : (F, \varphi_0, \varphi_2) \longrightarrow (\tilde{F}, \tilde{\varphi}_0, \tilde{\varphi}_2)$ between two tensor functors from $\mathcal{C}$ to $\mathcal{C}'$ is a natural transform. $\eta : F \longrightarrow \tilde{F}$ s.t. $\forall X, Y \in Ob(\mathcal{C})$ we have commutative diagrams

$$\begin{tikzcd} & F(I) \arrow[d, "\eta_I"] \\ I' \arrow[ur, "\varphi_0"] \arrow[r, "\tilde{\varphi}_0"'] & \tilde{F}(I) \end{tikzcd} \quad \text{and} \quad \begin{array}{ccc} F(X) \otimes' F(Y) & \xrightarrow{\varphi_2(X,Y)} & F(X \otimes Y) \\ \downarrow{\scriptstyle \eta_X \otimes' \eta_Y} & & \downarrow{\scriptstyle \eta_{X \otimes Y}} \\ \tilde{F}(X) \otimes' \tilde{F}(Y) & \xrightarrow{\tilde{\varphi}_2(X,Y)} & \tilde{F}(X \otimes Y) \end{array}$$

Furthermore, if $\eta_X$ is an isomorphism $\forall X \in Ob(\mathcal{C})$, then $\eta$ – natural $\otimes$ isomorphism

Finally, two tensor functors $F: \mathcal{C} \longrightarrow \mathcal{C}'$ and $G: \mathcal{C}' \longrightarrow \mathcal{C}$ together with natural tensor isomorphisms $id_{\mathcal{C}'} \xrightarrow{\simeq} FG$ and $GF \xrightarrow{\simeq} id_{\mathcal{C}}$ allow to define a tensor equivalence of categories