Lecture # 37

Goal for today: Discuss topological notions of knots, links, link diagrams, Reidemeister moves, Jones-Conway polynomial.

First, we shall define links and knots in $\mathbb{R}^3$. To this end, we start with:

Def: A *polygonal arc* L in $X=\mathbb{R}^3$ is a union of a finite number of segments
$$L=\bigcup_{i=1}^{n-1} \underbrace{[P_i, P_{i+1}]}_{\text{closed segment}}, \quad \text{such that } \underbrace{(P_i,P_{i+1})\cap(P_j,P_{j+1})}_{\text{open segments}}=\emptyset \text{ if } i\neq j.$$
Here: P_i – "vertices" of L, $[P_i,P_{i+1}]$ – "edges" of L.

The order on vertices defines an *orientation of L*, often indicated by arrows on its edges. For this reason P_1 is called "*origin of L*", P_n – "*endpoint of L*". We also will always consider only *simple* L, i.e. those polygonal arcs s.t. $\{P_1, P_2, \dots, P_{n-1}\}$ are pairwise distinct. On the other hand, if $P_1=P_n$ then L will be called "*closed*".

Def: A *link* L in $X=\mathbb{R}^3$ is the union of a finite number $\underset{\text{the order of } L}{m}$ of pairwise disjoint simple closed $\underbrace{\text{polygonal arcs}}_{\text{connected components of } L}$ in X. A *knot* is a link of order 1.

We shall consider links up to equivalence. In fact, there are two ways to define the latter (kind of discrete vs continuous) but they end up with the same equiv. rel-n.

Def: If L is a link in $X=\mathbb{R}^3$ and P_i, P_{i+1} are two consecutive vertices in a connected component of L, then given another point $Q\in X$ s.t.
$$Q\notin L,\ P_i\notin[Q,P_{i+1}],\ P_{i+1}\notin[Q,P_i], \text{ and } \operatorname{convex\,hull}(P_i,P_{i+1},Q)\cap L=[P_i,P_{i+1}]$$
define a new link L' obtained from L by a *Δ-operation*
$$L':=(L\setminus[P_i,P_{i+1}])\cup[P_i,Q]\cup[Q,P_{i+1}]$$
In other words, Δ-operations replace  when allowed.

Taking a symmetric & transitive closure of such operations defines $\sim_c$:

Def: Two links L & L' are *combinatorially equivalent*, denoted $L\sim_c L'$, if there is a sequence of links $L_0=L, L_1, L_2, \dots, L_{N-1}, L_N=L'$, such that $\forall\, 0\le i\le N-1$ one of $\{L_i, L_{i+1}\}$ is obtained from the other by a Δ-operation.

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On the other hand, there is a standard topological equivalence "up to isotopy".

Def: Two links L & L' are *isotopic*, denoted $L \underset{i}{\sim} L'$ if there is an orientation preserving isotopy h of $X = \mathbb{R}^3$ mapping L to L', i.e. a piece-wise linear map $h: [0,1] \times X \longrightarrow X$ s.t. $h(0,-) = id_X$, $h(t,-): X \to X$ - homeomorphism, and $h(1, L) = L'$.

Clearly, $\sim_i$ is an equivalence relation. Moreover, we have:

$\boxed{\text{Fact 1: } L \underset{c}{\sim} L' \iff L \underset{i}{\sim} L' \text{ for any two links } L, L' \text{ in } \mathbb{R}^3}$

hence, one can just write $L \sim L'$ omitting i,c.

Basic Problem: Given two links L, L' in $\mathbb{R}^3$ determine if they are equivalent or not

One way to approach this problem is to construct *invariants* for links $L \longmapsto I_L$, where I_L will be some algebraic object (number, polynomial, function, etc). Here, invariant means $L \sim L' \Rightarrow I_L = I_{L'}$, which shall allow to detect when L, L' are not equivalent

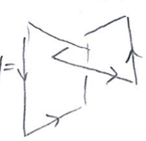
Two elementary examples:

(1) Order of a link ($= \#$ connected components)
This is clearly a link invariant. However, it's weak, e.g. all knots will have the same value $m=1$, but not every knot is equivalent to a trivial

(2) Linking number (either $m=2$, or treat every pair of connected components in L)
Given two connected components L_1, L_2 of a link L, one may define

$$\boxed{lk(L_1, L_2) = \frac{1}{2} \sum \varepsilon(P)}$$

where P runs through all crossings of L_1 & L_2 in link diagram, $\varepsilon(P) \in \{\pm 1\}$ is determined by the rule $\varepsilon(\ ) = 1$, $\varepsilon(\ ) = -1$.
defined on the next page.

Example: The Hopf link $H =$  is not trivial

Indeed $lk(H) = 1 \neq 0 = lk$ (trivial 2 component link )

The key goal for today is to introduce a much finer "Jones-Conway pol-l" invariant. But first, we shall discuss link diagrams that allow to think of links in 3 dim space rather as some diagrams on a plane.

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Def: a) A *link projection* Π is the union of a finite number of closed polygonal arcs in $\mathbb{R}^2$ s.t. no vertex is in an interior of another edge.

b) A *crossing point* of Π is a point lying in the interior of >1 edges of Π.

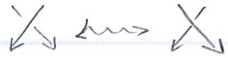
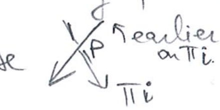
c) Π is *regular* if $\forall$ crossing point P, it lies in interior of exactly 2 edges of Π

d) Finally, a *link diagram* is a regular link projection Π in $\mathbb{R}^2$ s.t. $\forall$ crossing point P the set E_P, of 2 edges in interior of which P lies, is ordered.

The first edge in E_P is commonly called "overcrossing", 2^{nd} – "undercrossing". These are usually visualized by drawing  for overcrossing and  for undercrossing

Note: Any link diagram Π in $\mathbb{R}^2$ naturally gives rise to a link L in $\mathbb{R}^3$, defined up to isotopy

Lemma 1: Any link diagram Π may be turned to a link diagram of a trivial link in $\mathbb{R}^3$ by a sequence of "changes of crossings"

Here, changes of crossings means changing order of the set E_P at one of the crossing points P, so that . The above result is obvious: pick any polygonal arc Π_i of Π and start moving around it making all crossing points P into  unless the other edge was just travelled in which case 

After these changes $\forall i$, we shall get a link diagram of a trivial link

Notations: A trivial link of order m is denoted $O^{\otimes m} = \underbrace{OO\ldots O}_{m \text{ times}}$

In fact, not only every link diagram gives rise to a link in $\mathbb{R}^3$, but also any link gives rise to a family of link diagrams through projections $\mathbb{R}^3 \to \mathbb{R}^2$.

Fact 2: For any projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$, any link L is equivalent to a link L' such that $\pi(L')$ is a regular link projection (thus giving rise to a link diagram)

But now we get a new important question:

$\boxed{Q}$: When two link diagrams Π_1 & Π_2 represent equivalent links L_1 & L_2 (so that $L_1 \sim L_2$)?

The answer was provided almost a century ago by Reidemeister and requires the famous Reidemeister moves.

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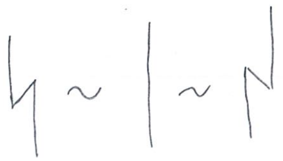
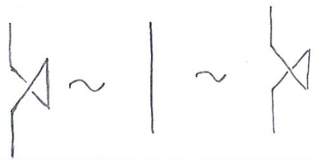
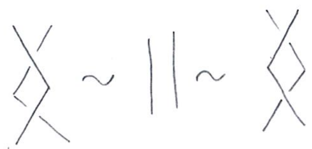
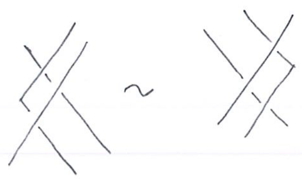
(R0) 

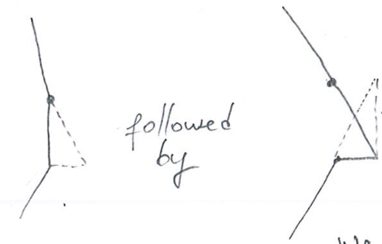
(R1) 

(R2) 

(R3) 

these moves can easily be seen as projections of Δ-operations on links hence, (R0)-(R3) do not change isotopy class of corresponding link

or



followed by

depending on whether this Δ is above or below we get two diagrams

depending on whether this Δ is above or below we get indicated 2 diagrams

Reidemeister moves, which allow to change a link diagram locally (without changing the rest)

Theorem of Reidemeister: Two generic link diagrams represent equivalent links in $\mathbb{R}^3$

$\Updownarrow$

they are related by a sequence of Reidemeister Moves (R1-R3) and isotopies of diagrams

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Here:

- a link diagram is called "generic" if any two of its vertices have different y-coord. (height)
  [clearly; moving each vertex slightly up/down any link diagram may be turned to generic]

- isotopy of link diagrams Π & Π' is an isotopy h of $\mathbb{R}^2$ s.t. $h(1,\Pi)=\Pi'$ and the orders of {Ep IP-crossing point} are preserved through isotopy h.
  [clearly, isotopic link diagrams represent isotopic links in $\mathbb{R}^3$]

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We are now ready to state & prove the main result for today, but we start with

Def: A triple (L_+, L_-, L_0) of oriented links in $\mathbb{R}^3$ is called a Conway triple if they can be represented by link diagrams (Π_+, Π_-, Π_0) that coincide outside a small disk in $\mathbb{R}^2$ and are isotopic inside this disk to:

 X_+ ,  X_- , $\downarrow\downarrow$ X_0.

Main Theorem (Jones-Conway polynomial invariant)

There is a unique map

$$\{\text{oriented links in } \mathbb{R}^3\} \longrightarrow \Lambda = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \quad \text{s.t.}$$
$$L \longmapsto P_L$$

1) if $L \sim L'$, then $P_L = P_{L'}$ (so that it's indeed invariant)

2) $P_O = 1$, where O = trivial knot

3) for any Conway triple (L_+, L_-, L_0) have
$$x \cdot P_{L_+} - x^{-1} P_{L_-} = y \, P_{L_0} \qquad (*)$$

P_L is called the Jones-Conway polynomial of a link L

$(*)$ are the famous skein relations

Remark: Various specializations of x, y provide some other invariants (not stronger than P_L though)

a) Specializing $x \mapsto 1$, $y \mapsto z$ get a 1-variable polynomial invariant $\nabla_L(z) = P_L(1, z)$ (in fact, $\nabla_L(z) \in \mathbb{Z}[z]$ not just $\mathbb{Z}[z^{\pm 1}]$)
Conway polynomial

b) Specializing $x \mapsto t^{-1}$, $y \mapsto t^{1/2} - t^{-1/2}$ yields $V_L(z) = P_L(t^{-1}, t^{1/2} - t^{-1/2}) \in \mathbb{Z}[t^{\pm 1/2}]$
Jones polynomial

Let's rephrase the statement of the theorem above. To this end, let's consider the set of equivalence classes of oriented links in $\mathbb{R}^3$, denoted by K, and define the Λ-module generated by K modulo skein relations

$$\boxed{\mathcal{V} := \Lambda[K] / \text{relations } (*)} \quad \leftarrow \text{"skein module of } \mathbb{R}^3\text{"}$$

The above is equivalent to:

Theorem 1: The Λ-linear map $Q: \Lambda \to \mathcal{V}$, $1 \mapsto \underset{}{[O]}$ (class of trivial knot), is an isomorphism

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The proof of surjectivity of Q is purely topological and follows from 2 lemmas:

Lemma 2: For any $n>1$, $[O^{\otimes n}] = \left(\frac{x-x^{-1}}{y}\right)^{n-1}[O]$ in V.

▸ Using the Conway triple () we conclude that in V:

$$x\cdot[O^{\otimes(n-1)}] - x^{-1}[O^{\otimes(n-1)}] = y\,[O^{\otimes n}] \Rightarrow [O^{\otimes n}] = \frac{x-x^{-1}}{y}[O^{\otimes(n-1)}] = \dots = \left(\frac{x-x^{-1}}{y}\right)^{n-1}[O]$$ ∎

Lemma 3: The Λ-module V is generated by $\{[O^{\otimes n}]\mid n\geq 1\}$

▸ The proof follows easily by induction on #{crossing points of a link diagram}, where one uses skein relation (*) together with Lemma 1, which tells that after some changes of crossings we shall get the trivial link ∎

Combining Lemmas 2-3, we get $Q:\Lambda\to V$ is surjective.

The proof of injectivity of Q requires the following result (to be proved in 2 classes)

Fact 3: For any $m\in\mathbb{Z}_{\geq 1}$ and $q\in\mathbb{C}^*\setminus\{\sqrt[?]{1}\}$, $\exists!$ map $\varphi_{m,q}:\{\text{oriented links in } \mathbb{R}^3\}\to\mathbb{C}$ s.t.

1) if $L\sim L'$, then $\varphi_{m,q}(L)=\varphi_{m,q}(L')$
2) $\varphi_{m,q}(O) = \frac{q^m-q^{-m}}{q-q^{-1}}$
3) for any Conway triple (L_+,L_-,L_0) have
$$q^m\varphi_{m,q}(L_+) - q^{-m}\varphi_{m,q}(L_-) = (q-q^{-1})\varphi_{m,q}(L_0)$$

Property 1) and 3) imply that $\varphi_{m,q}$ gives rise to Λ-linear map

$$\boxed{\varphi'_{m,q}: V\longrightarrow\mathbb{C}\ ,\quad f(x,y)\cdot[L]\longmapsto f(q^m, q-q^{-1})\cdot\varphi_{m,q}(L)}$$

by specializing $x\mapsto q^m$, $y\mapsto(q-q^{-1})$ and viewing $\mathbb{C}$ as a Λ-module under this

Now if $f(x,y)\in\Lambda$ is s.t. $Q(f)=0$, then $(\varphi'_{m,q}\circ Q)(f)=0$. But
$(\varphi'_{m,q}\circ Q)(f) = f(q^m, q-q^{-1})\cdot\frac{q^m-q^{-m}}{q-q^{-1}}$ & $q\neq\sqrt[?]{1}$ $\Big\}\Rightarrow$

$\Rightarrow f(q^m, q-q^{-1})=0$ $\forall q\in\mathbb{C}^*\setminus\{\sqrt[?]{1}\}$, $\forall m\in\mathbb{Z}_{\geq 1}$. This implies $f(x,y)\equiv 0$!

This completes proof of injectivity of Q, hence Thm 1, hence also Main Theorem ∎

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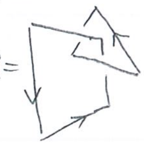
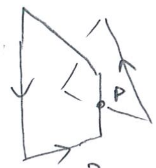
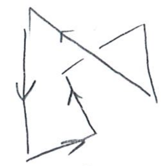
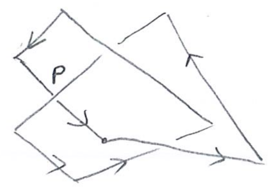
We conclude with a computation of two examples.

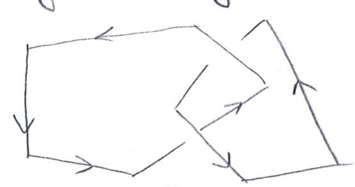
(1) Hopf link $H =$ 

Note the following Conway triple: ( $\cong H$,  $\cong O^{\otimes 2}$,  $\cong O$)

Hence: $x \cdot P_H - x^{-1} P_{O^{\otimes 2}} = y\, P_O \underset{\text{Lemma 2}}{\Longrightarrow} \boxed{P_H = x^{-1} \cdot y + (x^{-1} - x^{-3})\, y^{-1}}$

(2) Right-handed trefoil knot $K =$ 

In this case, we note the following Conway triple:

(K,  $\cong O$,  $\cong H$)

Hence: $x \cdot P_K - x^{-1} \cdot P_O = y \cdot P_H \overset{(1)}{\Longrightarrow} \boxed{P_K = x^{-2} y^2 + 2x^{-2} - x^{-4}}$

[Exercise: Verify that for the left-handed trefoil knot $\tilde{K}$: $P_{\tilde{K}} = 2x^2 - x^4 + x^2 y^2$.
Deduce that $K \not\sim \tilde{K}$