Lecture #38

Goal for today: Tangles, braids, and Artin's braid group.

Def: Given $k, l \in \mathbb{Z}_{\geq 0}$, a *tangle* L of type (k,l) is the union of a finite number of pairwise disjoint simple oriented polygonal arcs in $X = \mathbb{R}^2 \times [0,1]$ such that the boundary $\partial L = L \cap (\mathbb{R}^2 \times \{0,1\}) = \underbrace{[k]}_{\{1,2,\ldots,k\}} \times \{0\} \times \{0\} \amalg \underbrace{[l]}_{\{1,2,\ldots,l\}} \times \{0\} \times \{1\}$.

This generalizes the notion of links from last time: any link in $\mathbb{R}^2 \times [0,1]$ is a tangle of type $(0,0)$ (while any link in $\mathbb{R}^3$ is equivalent to a link in $\mathbb{R}^2 \times [0,1]$)

In what follows, we shall be primarily interested in oriented tangles. Given such an oriented tangle L of type (k,l), one defines two sequences of $\pm$:

$s(L) = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_i = +$ if $(i,0,0)$ - endpoint, $\varepsilon_i = -$ otherwise

$t(L) = (\eta_1, \ldots, \eta_l)$ with $\eta_j = +$ if $(j,0,1)$-origin, $\eta_j = -$ otherwise.

Example: $L =$  - tangle of type $(2,4)$ $\Rightarrow$ $s(L) = (+,+)$, $t(L) = (+,+,+,-)$

Some of the basic examples:

- $\downarrow$: straight line segment from $(1,0,1)$ to $(1,0,0)$ $\quad s(\downarrow) = + = t(\downarrow)$
- $\uparrow$: — // — $(1,0,0)$ to $(1,0,1)$ $\quad s(\uparrow) = - = t(\uparrow)$
- $\cap = $ ↗↘ - type $(2,0)$ tangle, e.g. line segment from $(1,0,0)$ to $(\frac{3}{2},0,\frac{1}{2})$, and then from $(\frac{3}{2},0,\frac{1}{2})$ to $(2,0,0)$ $\quad s(\cap) = (-+)$, $t(\cap) = \emptyset$
- $\overleftarrow{\cap} = $ ↙↖ - as above, but opposite orientation $\quad s(\overleftarrow{\cap}) = (+-)$, $t(\overleftarrow{\cap}) = \emptyset$
- $\cup = $ ↘↗ - type $(0,2)$ tangle $\quad s(\cup) = \emptyset$, $t(\cup) = (+-)$
- $\overleftarrow{\cup} = $ ↖↙ - type $(0,2)$ tangle $\quad s(\overleftarrow{\cup}) = \emptyset$, $t(\overleftarrow{\cup}) = (-+)$
- $\times = X_+$ - this can be realized by straightening ↙ towards us, ↘ outwards us, i.e. $(2,0,1)$ to $(\frac{3}{2},-\frac{1}{2},\frac{1}{2})$ followed to $(1,0,0)$ and $(1,0,1)$ to $(\frac{3}{2},\frac{1}{2},\frac{1}{2})$ followed to $(2,0,0)$ $\quad s(X_+) = (++) = t(X_+)$
- $\times = X_-$ - similar realization (swapping towards $\leftrightarrow$ outwards) $\quad s(X_-) = (--) = t(X_-)$

Equivalence relations

Completely analogously to the discussion of links from last time, we have:

1) combinatorial equivalence relation $\sim_c$ (transitive + symmetric closure of Δ-operations)
2) isotopy equivalence relation $\sim_i$ (w.r.t. similar notion of isotopy as last time)

$\boxed{\text{Fact 1: } L \sim_c L' \Leftrightarrow L \sim_i L' \text{ for any two tangles } L, L' \text{ in } \mathbb{R}^2 \times [0,1]}$. $\leftarrow$ henceforth, just use $\sim$

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Similarly to links, tangles can be represented by tangle diagrams:

Def: a) A tangle projection Π is the union of a finite number of polygonal arcs in $\mathbb{R}^2$ s.t. no vertex is in an interior of another edge.

b) A crossing point is a point sitting in the interior of >1 edges.

c) Π is regular if each crossing point P sits in the interior of exactly 2 edges.

d) Finally, a tangle diagram is a regular tangle projection in $\mathbb{R}\times[0,1]$ s.t. $\forall$ crossing point P the set E_P, of 2 edges in interior of which P lies, is ordered.

Alike link diagrams, we note that every (oriented) tangle diagram defines a tangle in $\mathbb{R}^2\times[0,1]$, unique up to isotopy. The opposite is also true:

Fact 2: For any projection $\pi: \mathbb{R}^2\times[0,1] \longrightarrow \mathbb{R}\times[0,1]$, any tangle L in $\mathbb{R}^2\times[0,1]$ is equivalent to a tangle L' such that $\pi(L')$ is a regular tangle projection.

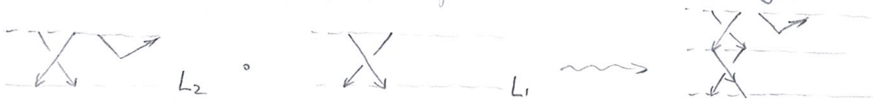
Finally, one has the following analogue of Reidemeister's Thm for links:

Fact 3: Two generic tangle diagrams represent equivalent tangles in $\mathbb{R}^2\times[0,1]$

$\Updownarrow$

they are related by a sequence of Reidemeister Moves (R1–R3) as well as isotopies of diagrams.

While all of this is completely analogous to what we had for links, there is one new operation we have for tangles – a composition of tangles. Namely, given tangles L_1, L_2 in $\mathbb{R}^2\times[0,1]$, one can place L_2 over L_1 by stacking up:



First, to be able to do this we need $k_2=\ell_1$ where L_i – tangle of type (k_i,ℓ_i), $i=1,2$.
And if tangles are oriented then we also need $s(L_2)=t(L_1)\in\{\pm1\}^{\ell_1=k_2}$

Finally, to make this a tangle in $\mathbb{R}^2\times[0,1]$ not $\mathbb{R}^2\times[0,2]$ we can zoom in intervals (up to isotopy doesn't matter how), e.g. we squeeze both L_1, L_2 twice in vertical direction by mapping $a_1: L_1 \longrightarrow \mathbb{R}^2\times I$, $(p,t)\longmapsto(p,\frac{t}{2})$, $a_2: L_2 \longrightarrow \mathbb{R}^2\times I$, $(p,t)\longmapsto(p,\frac{t}{2}+\frac{1}{2})$.

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Lemma 1: a) Composition of tangles is compatible with equivalence of tangles.
b) Composition is associative (up to equivalence)

a) Need to prove $L_2' \circ L_1' \sim L_2 \circ L_1$ if $L_2' \sim L_2$, $L_1' \sim L_1$.
But by Fact 3, tangles are equivalent if their diagrams are related by a sequence of isotopies and Reidemeister moves (R1–R3). But clearly performing such on L_1 and extending by identity on the top copy of L_2 (or vice versa), we get isotopies and Reidemeister moves of compositions. The result follows

b) The associativity is standardly established through isotopy

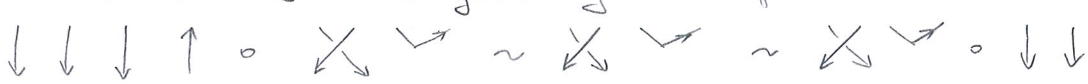
$(L_3 \circ L_2) \circ L_1 \quad \sim \quad L_3 \circ (L_2 \circ L_1)$

The isotopy will squeeze L_1 twice, extend L_3 twice, and shift L_2 by $1/4$, i.e.

$$h : t \times \mathbb{R}^2 \times \underbrace{[0,1]}_{z\text{-coordinate}} \longrightarrow \mathbb{R}^2 \times [0,1] \quad \text{via} \quad (t, \underset{\mathbb{R}^2}{p}, z) \longmapsto (p, \varphi_t(z)) \quad \text{where}$$

$$\varphi_t(z) = \begin{cases} z(1 - \frac{t}{2}) & \text{if } 0 \le z \le \frac{1}{2} \\ z - \frac{t}{4} & \text{if } \frac{1}{2} < z \le \frac{3}{4} \\ (1+t)z - t & \text{if } \frac{3}{4} < z \le 1 \end{cases}$$

Finally, we note by looking at the following example:



that for any tangle L, we always have $\boxed{id_{t(L)} \circ L \sim L \sim L \circ id_{s(L)}}$
where $id_{(\varepsilon_1, \dots, \varepsilon_k)}$ is a tangle of the segments from $(i,0,1)$ to $(i,0,0)$ $\forall\, 1 \le i \le k$ oriented down if $\varepsilon_i = +$ and up if $\varepsilon_i = -$.

Remark: One can also set $L_2 \circ L_1 = 0$ if they are not stackable, set $id = \sum_k \sum_{\varepsilon_1, \dots, \varepsilon_k \in \pm} id_{(\varepsilon_1, \dots, \varepsilon_k)}$ – infinite sum $\Rightarrow$ id will be an identity

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A special interesting class of tangles are braids:

Def: A braid L with n strands is a tangle of type (n,n) such that

1) $s(L) = (\underbrace{+, \ldots, +}_{n \text{ times}}) = t(L)$
2) L has no closed arcs
3) $L \cap (\mathbb{R}^2 \times \{z\})$ consists of n points $\forall z \in [0,1]$.

this means that L has no local extremums w.r.t. height

One can repeat all the previous constructions in the present setup, namely:

- equivalence relation of braids ($\sim_c = \sim_i$)
- represent each braid by a braid diagram
- isotopy relation on braid diagrams

One has the following analogue of Fact 3 for the class of $\{\text{braids}\} \subset \{\text{tangles}\}$:

> **Fact 4:** Two braid diagrams represent equivalent braids
>
> $\Updownarrow$
>
> they are related by a sequence of Reidemeister moves (R2, R3) as well as isotopies of diagrams

note that (R1) is forbidden!

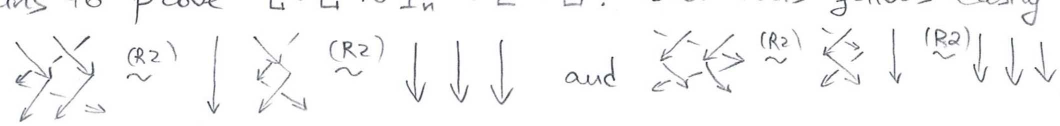
Since $s(L) = t(L) = (+, \ldots, +)$ is the same for all braids with n strands, their composition is always defined, see Lemma 1. Moreover, $id_{(+,\ldots,+)} =: 1_n$ depicted by $\overset{1}{\underset{1}{\downarrow}} \cdots \overset{n}{\underset{n}{\downarrow}}$ is clearly a unit, i.e. $1_n \circ L \sim L \sim L \circ 1_n$.

Finally, one has one more operation $L \rightsquigarrow L^{-1}$ – "inverse braid" obtained by reflecting L in the plane $\mathbb{R}^2 \times \{\frac{1}{2}\}$ (but oriented downwards). For example

Let B_n be the set of equivalence classes of braids with n strands. Then:

> **Lemma 2:** $\circ, 1_n, L \mapsto L^{-1}$ induce a group structure on B_n ← "(Artin's) Braid group"

▸ Remains to prove $L \circ L^{-1} \sim 1_n \sim L^{-1} \circ L$. But this follows easily by a sequence of (R2) moves, e.g.



∎

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As we shall see now, this braid group B_n is indeed the same as the one from Lecture 29.
To this end, let's consider the following special braids

$\sigma_i = \underbrace{\downarrow \cdots \downarrow}_{i-1} X_+ \underbrace{\downarrow \cdots \downarrow}_{n-i-1}$ with $X_+ =$, so that $\sigma_i^{-1} = \underbrace{\downarrow \cdots \downarrow}_{i-1} X_- \underbrace{\downarrow \cdots \downarrow}_{n-i-1}$ with $X_- =$

The following is easy:

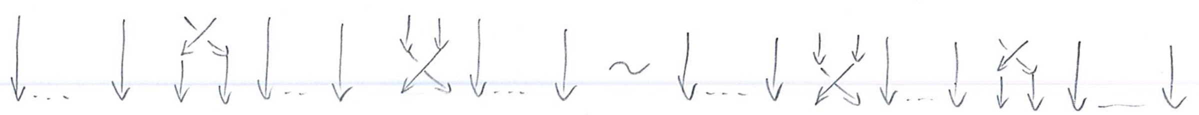
Lemma 3: a) B_n is generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$ (and their inverses)

b) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j|>1$

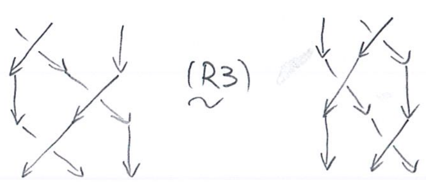
$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

▷ a) Draw a braid diagram for a braid and slice it by several horizontal lines into a composition of braid diagrams each containing 1 crossing point, hence equivalent to σ_i or σ_i^{-1} for some i.

b) The first relation is clear (two diagrams are equivalent up to isotopy):



The second relation follows by performing (R3) move



In fact, we have the following theorem of Artin:

Theorem 1: The group B_n is isomorphic to the group generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$ with the defining relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j|>1$

↑ this is how we introduced B_n in the Remark of Lecture 29.

Remark: Note that $B_n \twoheadrightarrow S_n$, where given a braid L with n strands we define a permutation $L \longmapsto \sigma_L \in S_n$ that maps $k \mapsto \sigma_L(k)$ so that the points $(\sigma_L(k), 0, 1)$ & $(k, 0, 0)$ are in the same component of L. The kernel $P_n := \mathrm{Ker}(B_n \twoheadrightarrow S_n)$ is called the *pure braid group*

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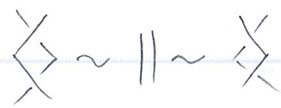
Proof of Theorem 1

Let G be the group generated by $n-1$ el-s $\{\sigma_1, \ldots, \sigma_{n-1}\}$ with the defining rel-s
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j=\sigma_j\sigma_i \ \text{ if } |i-j|>1.$$

Suffices: Construct a group homomorphism $\underset{\sigma_i \mapsto \sigma_i^{\psi}}{B_n \to G}$

Given any braid L, draw its braid diagram and slightly move crossing points (by isotopy equiv) so that all of them have different height, and then slice it by horizontal lines to represent as a composition of braid diagrams with 1 crossing point each. This allows to find a braid equivalent to L, written as a product of $\sigma_i^{\pm 1}$. Hence, to such L we associate the same-named product of el-s in G, call $\rho(L)$.

It remains to show that this construction of $\rho(L)$ is independent for different choices of decomposition $L \sim \sigma_{i_1}^{\pm 1} \cdots \sigma_{i_N}^{\pm 1}$. But according to Fact 4, it suffices to check that (R2) move, (R3) move, and isotopies give rise to the same elts $\in G$.

- (R2) move :  the corresponding words $\rho(L)$ differ by changing the product $\sigma_i\sigma_i^{-1}$ by 1 or $\sigma_i^{-1}\sigma_i$, hence, clearly results in the same $\rho(L)\in G$.
- (R3) move :  replaces $\sigma_i\sigma_{i+1}\sigma_i$ by $\sigma_{i+1}\sigma_i\sigma_{i+1}$, hence, results in the same $\rho(L)\in G$ as we had rel-n $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$
- performing isotopies can be broken up in smaller steps which either don't change $\rho(L)$ at all or change the order of crossing pts apart (w.r.t. height) which amounts to replacing $\sigma_i^{\varepsilon}\sigma_j^{\varepsilon'}$ by $\sigma_j^{\varepsilon'}\sigma_i^{\varepsilon}$ for $|i-j|>1$ and any $\varepsilon, \varepsilon' \in \{\pm 1\}$. But this still gives the same elt in G by defining rel-s $\sigma_i\sigma_j=\sigma_j\sigma_i$ $(|i-j|>1)$ in G. ∎

To relate it back to Lecture 29 and its generalization, we note:

> **Lemma 4**: Given a vector space V, $c\in \mathrm{End}(V^{\otimes 2})$ satisfying $\underset{\text{quantum Yang-Baxter}}{\text{q.YB}}$-equation $C_{12}C_{23}C_{12}=C_{23}C_{12}C_{23}$ gives rise to an action of the braid gp $B_n \curvearrowright V^{\otimes n}$ via $\sigma_i \mapsto C_{i,i+1}$

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Finally, let's provide yet another topological realization of the braid groups.

Def: For $n \geq 1$, let $Y_n = (\mathbb{C}^n)^{reg} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}$ – open subset of $\mathbb{C}^n$.
The configuration space of n distinct points in $\mathbb{C}$ is $X_n := Y_n / S_n$, endowed with the quotient topology.

Theorem 2: a) $B_n := \pi_1(X_n) = \pi_1(X_n, \underbrace{\{1, \dots, n\}}_{\text{viewed as a point in } X_n})$

b) $P_n := \pi_1(Y_n) = \pi_1(Y_n, \underbrace{(1, \dots, n)}_{\text{viewed as a point in } Y_n})$

Here, $\pi_1(X, p)$ denotes the fundamental group of a topological set X w.r.t. "base" point $p \in X$ (the choice of p is irrelevant, but the above ones are to match nicely with braids). In other words, $\pi_1(X, p)$ consists of homotopy classes of loops $f : [0,1] \to X$ s.t. $f(0) = p = f(1)$.

In our context, we work with piecewise linear maps f only. The composition is obtained by concatenating such loops (and scaling time by 2).

In the case of $X = X_n$, $p = \{1, \dots, n\} \in X$, a loop is nothing else than a piecewise linear map $f = (f_1, \dots, f_n) : [0,1] \to \mathbb{C}^n$ s.t. $f(0) = (1, \dots, n)$ and $\{f_1(1), \dots, f_n(1)\} = \{1, 2, \dots, n\}$ as sets, while $f_i(t) \neq f_j(t)$ $\forall i \neq j$ $\forall t \in [0,1]$.
With this perspective in mind, given a loop in X_n, one obtains a braid with n strands by taking $\bigcup_{i=1}^{n} \mathrm{Graph}(f_i)$, where $[0,1]$ plays the role of z-coord.
Vice versa, given a braid L with n strands, we define $f_i(t)$ to be the projection on $\mathbb{R}^2_{xy} = \mathbb{C}$ of the intersection $(\mathbb{R}^2 \times \{t\}) \cap \left\{\begin{matrix}\text{Connected cpt of } L \\ \text{ending at } (i,0,0)\end{matrix}\right)$ which defines a loop $f = (f_1, \dots, f_n)$.

It's straightforward to see that:

- two braids are equivalent $\iff$ corresponding loops are isotopic in X_n
- the product in $\pi_1(X_n, p)$ matches the composition of braids as well as taking inverse and the unit elts

This implies part a) of Thm. Part b) is very similar ← Exercise!