Lectures #39–40

Goal: Tangle category, its presentation by generators & relations, applications

We shall start with the notion of strict $\otimes$-categories, strict $\otimes$-functors, cf. Lecture 36.

Def: The tensor category $(\mathcal{C},\otimes,I,a,l,r)$ is strict if a,l,r are all identities.

What that means is that: $\begin{cases} I\otimes A=A,\ A\otimes I=A,\ (A\otimes B)\otimes C=A\otimes(B\otimes C) & \forall A,B,C\in Ob(\mathcal{C}) \\ Id_I\otimes f=f,\ f\otimes Id_I=f,\ (f\otimes g)\otimes h=f\otimes(g\otimes h) & \forall f,g,h\in Hom(\mathcal{C}) \end{cases}$

Example 1: Consider a family of groups $\{G_k\}_{k\in\mathbb{Z}_{\geq 0}}$ such that $G_0=\{e\}$.

Define a category $\mathcal{G}$ s.t. $Ob(\mathcal{G})=\mathbb{Z}_{\geq0}$, $Hom_{\mathcal{G}}(k,l)=\begin{cases}\emptyset, & k\neq l\\ G_k, & k=l\end{cases}$

The composition is a group product. We would like to make $\mathcal{G}$ into a strict $\otimes$-category with $n\otimes m=n+m$ – on the level of obj-s.

To this end, we need a family of group homomorphisms

$$\rho_{n,m}: G_n\times G_m\longrightarrow G_{n+m}\quad \text{s.t. } \rho_{n,0}=id_{G_n}=\rho_{0,n}$$

which will allow us to tensor morphisms:

$$(n\xrightarrow{f}n)\otimes(m\xrightarrow{g}m):=(n+m)\xrightarrow{\rho_{n,m}(f,g)}(n+m)$$

[Exercise: This makes $\mathcal{G}$ into a strict $\otimes$-category iff

$$\rho_{n+m,k}\circ(\rho_{n,m}\otimes id_{G_k})=\rho_{n,m+k}\circ(id_{G_n}\otimes\rho_{m,k})\quad\forall n,m,k\in\mathbb{Z}_{\geq0}$$

The two prototypical examples of this construction are:

[A] A family $G_n=GL_n(\mathbb{k})$ $\forall n\geq1$, $G_0=\{e\}$, with $\rho_{n,m}(f,g)=\left(\begin{array}{c|c} f & 0\\ \hline 0 & g\end{array}\right)$

$\rightsquigarrow$ the resulting category is GL.

[B] A family $G_n=\underbrace{S_n}_{\text{symmetric gp}}$ $\forall n\geq1$, $G_0=\{e\}$ with $\rho_{n,m}(\sigma,\tau)=\underbrace{\begin{matrix}1\dots n\\ \downarrow\sigma\\ 1\dots n\end{matrix}}_{\sigma}\ \underbrace{\begin{matrix}n+1\dots n+m\\ \downarrow\ \downarrow\ \swarrow\\ n+1\dots n+m\end{matrix}}_{\tau}$

or equivalently we can view S_n as a subgp of $GL_n(\mathbb{k})$ and restrict $\rho_{n,m}$ from the previous example $\rightsquigarrow$ the resulting category is $\mathcal{S}$.

Def: A tensor functor (F,φ_0,φ_2) from a tensor category $\mathcal{C}$ to $\mathcal{C}'$, see Lecture 36, is strict, if $\varphi_0: I'\to F(I)$ and $\varphi_2(X,Y): F(X)\otimes' F(Y)\to F(X\otimes Y)$ are identities $\forall X,Y\in Ob(\mathcal{C})$

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Another important example of strict $\otimes$-categories is the tangle category:

Example 2: The tangle category $\mathcal{T}$.

Define $Ob(\mathcal{T}) = \{$finite sequences of $\pm$'s$\}$, define $I :=$ empty sequence $\emptyset \in Ob(\mathcal{T})$

$Hom(\mathcal{T}) = \{$isotopy classes of oriented tangles$\}$, see Lecture #38

Here: 1) the composition of morphisms is just the composition of tangles (well-defined due to Lemma 1 of Lecture 38)

2) the two maps $Hom(\mathcal{T}) \underset{s,t}{\rightrightarrows} Ob(\mathcal{T})$ associate to any tangle L the sequences $s(L), t(L)$ as defined in Lect 38

(e.g. the class of  is a morphism $++ \to +++-$)

3) for any object $\underline{\varepsilon} = (\varepsilon_1, \_\_, \varepsilon_k) \in \{\pm\}^k$ - object in $\mathcal{T}$, the identity $id_{\underline{\varepsilon}}$ is an isotopy class of the tangle

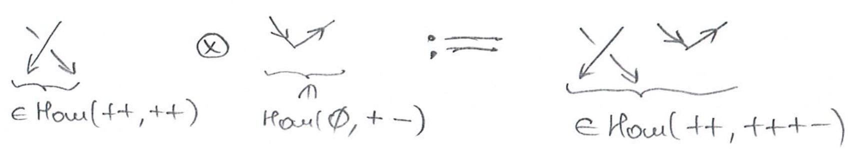
where i^{th} line segment is oriented down if $\varepsilon_i = +$, up if $\varepsilon_i = -$

This defines a category $\mathcal{T}$. However, the point is that it's actually a strict $\otimes$-category. On objects, $\otimes$ is just a concatenation:

$$(\varepsilon_1, \_\_, \varepsilon_k) \otimes (\varepsilon_1', \_\_, \varepsilon_\ell') := (\varepsilon_1, \_\_, \varepsilon_k, \varepsilon_1', \_\_, \varepsilon_\ell') \quad \forall k, \ell \geq 1$$

$$\emptyset \otimes (\varepsilon_1, \_\_, \varepsilon_k) := (\varepsilon_1, \_\_, \varepsilon_k) =: (\varepsilon_1, \_\_, \varepsilon_k) \otimes \emptyset$$

On morphisms, $\otimes$ is just stacking left-to-right of two tangles, e.g.



Clearly this is well-defined up to isotopy, associative on isotopy classes:

$$\boxed{\mathcal{T} \text{ is a strict tensor category}}$$

[**Remark**: $End(I) = End(\emptyset) = \{$links in $\mathbb{R}^2 \times (0,1)\} /$isotopy

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The key result for today is the realization of $\mathcal{T}$ by generators & relations. But even to state the result, we need to develop this general formalism.

Input: A strict $\otimes$-category $(\mathcal{C}, \otimes, I)$ together with a collection $\mathcal{F} \subseteq \mathrm{Hom}(\mathcal{C})$.

Def: We define words w in $\mathcal{F}$ together with associating $\overline{w} \in \mathrm{Hom}(\mathcal{C})$ as follows:

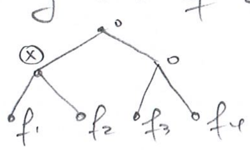
1) a word of rank ≤ 1 is either a symbol $[f]$ (with $f \in \mathcal{F}$) or $[\mathrm{id}_V]$ ($V \in \mathrm{Ob}\,\mathcal{C}$) and associated morphisms are $\overline{[f]} = f$, $\overline{[\mathrm{id}_V]} = \mathrm{id}_V$.

2) define words of rank $\leq n+1$ as formal symbols of the form
$u \otimes v$ or $u \circ v$ (assuming $\overline{u} \circ \overline{v}$ is well-defined) where u, v are words of rank $\leq n$
and associated morphisms are $\overline{u \otimes v} = \overline{u} \otimes \overline{v}$, $\overline{u \circ v} = \overline{u} \circ \overline{v}$.

Then, the words in $\mathcal{F}$ is the union of all words of rank $\leq n$ $\forall n \geq 1$.

Remark: One can interpret this as binary trees, whose leaves are labeled by elts of $\mathcal{F}$, while each internal vertex is labeled $\otimes$ or $\circ$

e.g.



$\rightsquigarrow ([f_1] \otimes [f_2]) \circ ([f_3] \circ [f_4])$

Def: Two words w, w' in $\mathcal{F}$ are equivalent, $w \sim w'$, if there is a sequence of words $w = w_1, w_2, \ldots, w_{N-1}, w_N = w'$ s.t. $\forall\ 1 \leq i < N$ the word w_{i+1} is obtained from w_i by replacing a subword by another from the following list:

1) $([f] \circ [g]) \circ [h] \sim [f] \circ ([g] \circ [h])$
2) $[\mathrm{id}_{t(f)}] \circ [f] \sim [f]$, $[f] \circ [\mathrm{id}_{s(f)}] \sim [f]$
3) $[\mathrm{id}_V] \circ [\mathrm{id}_V] \sim [\mathrm{id}_V]$
4) $([f] \otimes [g]) \otimes [h] \sim [f] \otimes ([g] \otimes [h])$
5) $[\mathrm{id}_I] \otimes [f] \sim [f]$, $[f] \otimes [\mathrm{id}_I] \sim [f]$
6) $[\mathrm{id}_X] \otimes [\mathrm{id}_Y] \sim [\mathrm{id}_{X \otimes Y}]$
7) $([f] \otimes [g]) \circ ([f'] \otimes [g']) \sim ([f \circ f']) \otimes ([g \circ g'])$

$\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$
$f, g, f', g', h \in \mathcal{F}$.

Note: $w \sim w' \Rightarrow \overline{w} = \overline{w'} \in \mathrm{Hom}(\mathcal{C})$

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The following is straightforward:

Lemma 1: a) $\forall f, g \in \mathcal{F}$:

$$([f]\otimes[id_{t(g)}])\circ([id_{s(f)}]\otimes g) \sim ([id_{t(f)}]\otimes[g])\circ([f]\otimes[id_{s(g)}])$$

b) $\forall f_1, \ldots, f_k \in \mathcal{F}$ (s.t. $s(f_i) = t(f_{i+1})$ $\forall 1\le i<k$), $\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$:

$$([id_X]\otimes[f_1]\otimes[id_Y])\circ\ldots\circ([id_X]\otimes[f_k]\otimes[id_Y]) \sim [id_X]\otimes([f_1]\circ\ldots\circ[f_k])\otimes[id_Y].$$

c) Any word in $\mathcal{F}$ is equivalent to a word $[id_X]$ or to

$$([id_{X_1}]\otimes[f_1]\otimes[id_{Y_1}])\circ\ldots\circ([id_{X_k}]\otimes[f_k]\otimes[id_{Y_k}])$$

a) $([f]\otimes[id_{t(g)}])\circ([id_{s(f)}]\otimes[g]) \overset{7)}{\sim} ([f]\circ[id_{s(f)}])\otimes([id_{t(g)}]\circ[g]) \overset{2)}{\sim} [f]\otimes[g]$

Likewise: $([id_{t(f)}]\otimes[g])\circ([f]\otimes[id_{s(g)}])$ $\overset{7)+2)}{\sim}$ ✓

b) same argument (can do by induction on k)

c) Argue by induction on a rank. For words of rank ≤ 1 it's clear as

$$[f] \overset{5)}{\sim} [id_I]\otimes[f]\otimes[id_I]$$

For the step of induction, the only nontrivial case is when a word w of rank $\le n+1$ is written as $w = u\otimes v$ where by hypothesis

$$u \sim u_1\circ\ldots\circ u_k \quad \text{with} \quad u_i \sim [id_{X_i}]\otimes[f_i]\otimes[id_{Y_i}]$$

$$v \sim v_1\circ\ldots\circ v_\ell \qquad\qquad v_j \sim [id_{X'_j}]\otimes[f'_j]\otimes[id_{Y'_j}].$$

Then:

$$w \underset{7),2)}{\sim} (u_1\otimes[id_T])\circ\ldots\circ([u_k]\otimes[id_T])\circ([id_S]\otimes[v_1])\circ\ldots\circ([id_S]\otimes[v_\ell])$$

where $T = t(\overline{v}_1)$, $S = s(\overline{u}_k)$

But according to 6), we have:

$$\left.\begin{array}{l}[u_i]\otimes[id_T] \sim [id_{X_i}]\otimes[f_i]\otimes[id_{Y_i\otimes T}] \\ [id_S]\otimes[v_j] \sim [id_{S\otimes X'_j}]\otimes[f'_j]\otimes[id_{Y'_j}]\end{array}\right\} \Rightarrow \text{completes induction step}$$

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Notation: Let $\mathcal{M}(\mathcal{F})$ be the class of equivalence classes of words in $\mathcal{F}$.

Note that since $\bar{a}=\bar{a'}$ whenever $a\sim a'$, we have a well-defined $\bar{a}\in \mathrm{Hom}\,\mathcal{C}$ $\forall a\in\mathcal{M}(\mathcal{F})$.

Def: Define a strict tensor category $\mathcal{C}(\mathcal{F})$ as follows:

1) $\mathrm{Ob}\,\mathcal{C}(\mathcal{F})=\mathrm{Ob}\,\mathcal{C}$
2) $\mathrm{Hom}\,\mathcal{C}(\mathcal{F})=\mathcal{M}(\mathcal{F})$
3) composition of words – part of their definition
4) $\otimes$-product of objects – as in $\mathcal{C}$
5) $\otimes$-product of words – part of their def-n
6) $s,t:\mathrm{Hom}\,\mathcal{C}(\mathcal{F})\to\mathrm{Ob}\,\mathcal{C}(\mathcal{F})$ are $s(a)=s(\bar{a})$, $t(a)=t(\bar{a})$
7) $\forall X\in\mathrm{Ob}\,\mathcal{C}=\mathrm{Ob}\,\mathcal{C}(\mathcal{F})$, the class of the word $[id_X]$ is the identity morphism

Note: Always have a strict tensor functor

$$\boxed{\mathcal{C}(\mathcal{F})\longrightarrow\mathcal{C}\quad\text{sending}\quad \underbrace{X\longmapsto X}_{\text{objects}}\ ,\ \underbrace{a\longmapsto\bar{a}}_{\text{morphisms}}}$$

Def: A collection $\mathcal{F}$ generates $\mathcal{C}$ if any morphism in $\mathcal{C}$ is of the form $\bar{a}$ for $a\in\mathcal{M}(\mathcal{F})$

[In other words, any morphism can be obtained from $\{f\}_{f\in\mathcal{F}}\cup\{[id_X]\}_{X\in\mathrm{Ob}\,\mathcal{C}}$ via $\circ$, $\otimes$.

Def: A strict $\otimes$-category $\mathcal{C}$ is free in a collection $\mathcal{F}$ if $\mathcal{F}$ generates $\mathcal{C}$ and

$$a\sim a'\Longleftrightarrow \bar{a}=\bar{a'}\qquad\forall\text{ words } a,a' \text{ in } \mathcal{F}.$$

[Evoking Prop. 1 of Lecture #35, we see that $\mathcal{C}$ is free in $\mathcal{F}$ iff $\mathcal{C}(\mathcal{F})\to\mathcal{C}$ is equivalence of categories

However, similarly to groups, algebras, Lie algebras, it's rare to have free obj-s, and one rather needs a notion of "being generated by generators & rel-s".

Def: Given a collection R of pairs (u,v) of words in $\mathcal{F}$ s.t. $\bar{u}=\bar{v}\in\mathrm{Hom}\,\mathcal{C}$, we define a "congruence $\sim_R$" equivalence relation on $\mathcal{M}(\mathcal{F})$:

$a\underset{R}{\sim}b\Longleftrightarrow$ there is a sequence $a=a_0,a_1,\dots,a_{N-1},a_N=b$ such that $\forall i$ the words a_i,a_{i+1} are obtained one from another by replacing a subword u by v for some $(u,v)\in R$

Def: $\mathcal{C}$ is generated by $\mathcal{F}$ and relations R if $\mathcal{C}$ is generated by $\mathcal{F}$ and

$$\bar{a}=\bar{a'}\Longleftrightarrow a\sim_R a'\quad\forall a,a'\in\mathcal{M}(\mathcal{F})$$

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Remark: In algebra, one of the key benefits of an explicit realization by "generators & relations" is that it allows to describe morphisms from that object

An analogue of this remark for categories is the following simple observation:

Lemma 2: Let $\mathcal{C}$ – strict $\otimes$-category generated by $\mathcal{F} \subset \mathrm{Hom}\,\mathcal{C}$ and relations $\mathcal{R}$. Let's also be given another strict $\otimes$-category $\mathcal{D}$ together with

1) map $F_0 : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ s.t. $F_0(I) = I$, $F_0(X \otimes X') = F_0(X) \otimes F_0(X')$ $\forall X, X' \in \mathrm{Ob}\,\mathcal{C}$
2) $\forall f \in \mathcal{F}$ have a morphism $g_f : F_0(s(f)) \to F_0(t(f))$

Then: $\exists!$ strict $\otimes$-functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(X) = F_0(X)$, $F(f) = g_f$ $\forall X \in \mathrm{Ob}\,\mathcal{C}, f \in \mathcal{F}$

$\Updownarrow$

for any $(u, v) \in \mathcal{R}$ we get the same morphisms in $\mathcal{D}$ after replacing any subword $[f] \rightsquigarrow [g_f]$, $[\mathrm{id}_X] \rightsquigarrow [\mathrm{id}_{F_0(X)}]$

Exercise: Prove this lemma (it's [Kassel, Prop XII.1.4])

We shall be now applying the above concept in the case of the tangle category $\mathcal{T}$ from page 2. To this end we recall the following specific tangles:

Also since $\otimes$ in $\mathcal{T}$ is just stacking of tangles left-to-right we shall simply write $L L'$ instead of $L \otimes L'$.

Remark: Similarly to the proof of Lemma 3 from Lecture #38, we note that any generic tangle diagram can be sliced by several horizontal lines into a composition of tangle diagrams with either 1 local min/max point or 1 crossing point

We are now ready to state the key result for today.

$\downarrow$

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Theorem 1: The strict $\otimes$-category $\mathcal{T}$ is generated by six morphisms
$$\cup,\ \overleftarrow{\cup},\ \cap,\ \overleftarrow{\cap},\ X_+,\ X_-$$
and the following relations

1) $(\downarrow \cap)\circ(\cup\downarrow) = \downarrow = (\overleftarrow{\cap}\downarrow)\circ(\downarrow\overleftarrow{\cup})$

2) $(\uparrow\overleftarrow{\cap})\circ(\overleftarrow{\cup}\uparrow) = \uparrow = (\cap\uparrow)\circ(\uparrow\cup)$

3) $(\cap\uparrow\uparrow)\circ(\uparrow\cap\downarrow\uparrow\uparrow)\circ(\uparrow\uparrow X_\pm\uparrow\uparrow)\circ(\uparrow\uparrow\downarrow\cup\uparrow)\circ(\uparrow\uparrow\cup)$
$$\|$$
$(\uparrow\uparrow\overleftarrow{\cap})\circ(\uparrow\uparrow\downarrow\overleftarrow{\cap}\uparrow)\circ(\uparrow\uparrow X_\pm\uparrow\uparrow)\circ(\uparrow\overleftarrow{\cup}\downarrow\uparrow\uparrow)\circ(\overleftarrow{\cup}\uparrow\uparrow)$

4) $X_+\circ X_- = \downarrow\downarrow = X_-\circ X_+$

5) $(X_+\downarrow)\circ(\downarrow X_+)\circ(X_+\downarrow) = (\downarrow X_+)\circ(X_+\downarrow)\circ(\downarrow X_+)$

6) $(\downarrow\overleftarrow{\cap})\circ(X_\pm\uparrow)\circ(\downarrow\cup) = \downarrow$

7) $(\cap\downarrow\uparrow)\circ(\uparrow X_\mp\uparrow)\circ(\uparrow\downarrow\cup)\circ(\uparrow\downarrow\overleftarrow{\cap})\circ(\uparrow X_\pm\uparrow)\circ(\overleftarrow{\cup}\downarrow\uparrow) = \downarrow\uparrow$

8) $(\uparrow\downarrow\overleftarrow{\cap})\circ(\uparrow X_\pm\uparrow)\circ(\overleftarrow{\cup}\downarrow\uparrow)\circ(\cap\downarrow\uparrow)\circ(\uparrow X_\mp\uparrow)\circ(\uparrow\downarrow\cup) = \uparrow\downarrow$

It's very instructive to check these rel-s hold in $\mathcal{T}$ by drawing the tangles (we did this in class, but in the notes just refer to [Kassel, p. 301])

In particular:

1)–3) – just isotopies

4), 7), 8) – (R2) moves + isotopies

5) – (R3) moves + isotopies

6) – (R1) moves + isotopies

Proof: Read [Kassel, Sect. XII.3] who in turn borrowed exposition from [Turaev]

But the idea is quite clear: split equivalence of two tangles into several Reidemeister moves + simple isotopies of the corresponding tangle diagrams and verify that each of these "basic transformations" doesn't change the congruence class of words.

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As noted earlier (page 6), the presentation by generators & rels allows to construct functors to other strict categories. In particular, one thus obtains invariants of links. Indeed, any link in $\mathbb{R}^3$ is isotopic to a link in $\mathbb{R}^2\times(0,1)$, while the isotopy classes of the latter are precisely $\mathrm{End}_{\mathcal{T}}(\underset{\uparrow}{I})$
unit object in $\mathcal{T}$ which is $\emptyset$.

So: Any tensor functor

$$F:\mathcal{T}\longrightarrow\mathcal{D}$$

associates to a link L an endomorphism $F(L)\in \mathrm{End}_{\mathcal{D}}(I)$ (unit object in $\mathcal{D}$).

! It's explained in [Kassel, XII.4-5] how to obtain the map

$$\varphi_{m,q}:\{\text{oriented links in } \mathbb{R}^3\}/\text{isotopy}\longrightarrow\mathbb{C}$$

from [Fact 3, Lecture #37] that way. (with $\mathcal{D}=(\mathrm{Vect}^{f.d.}_{\Bbbk})^{str}$ - see discussion below)

While we don't have time to work out all the details, let's conclude this note by explaining how to get a strict $\otimes$-category $\mathcal{C}^{str}$ from just a $\otimes$-category $\mathcal{C}$.

$\mathcal{C}\rightsquigarrow\mathcal{C}^{str}$ (for any $\otimes$-category $\mathcal{C}$ we construct $\otimes$-equivalent strict $\otimes$-category $\mathcal{C}^{str}$)

Our goal is to construct a strict tensor category $\mathcal{C}^{str}$, tensor equivalent to $\mathcal{C}$.

- $\mathrm{Ob}(\mathcal{C}^{str})=\{\text{all finite sequences of objects in } \mathcal{C}\}=\{(X_1,\dots,X_k)\mid {k\geq 0 \atop X_1,\dots,X_k\in \mathrm{Ob}\,\mathcal{C}}\}$

  note that $k=0$ corresponds to an empty sequence $\emptyset$
- Construct a map $F:\mathrm{Ob}(\mathcal{C}^{str})\longrightarrow \mathrm{Ob}(\mathcal{C})$ as follows:

  $F(\emptyset)=I$ - unit of $\mathcal{C}$, $F(\underbrace{X_1}_{\text{length 1 sequences}})=X_1$, $F(X_1,\dots,X_k)=(\dots((X_1\otimes X_2)\otimes X_3)\otimes\dots)\otimes X_k$
- $\mathrm{Hom}_{\mathcal{C}^{str}}(S,S'):=\mathrm{Hom}_{\mathcal{C}}(F(S),F(S'))$ where S,S' are sequences as above:

  $S=(X_1,\dots,X_k)$, $S'=(X'_1,\dots,X'_\ell)$.

Note that composition of morphisms in $\mathcal{C}$ thus gives rise to a composition of morphisms in $\mathcal{C}^{str}$. Also, $\mathrm{id}_{F(S)}$ in $\mathcal{C}$ gives rise to id_S in $\mathcal{C}^{str}$. Thus, we obtain a category $\mathcal{C}^{str}$, equipped with a functor $F:\mathcal{C}^{str}\longrightarrow\mathcal{C}$ (on objects - defined above, on morphisms - identity).

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(Continuation of $\mathcal{C} \rightsquigarrow \mathcal{C}^{str}$)

As $F: \mathcal{C}^{str} \to \mathcal{C}$ is clearly fully faithful and essentially surjective, we get:

$\boxed{\text{Lemma 3: a) } \mathcal{C}^{str} \xrightarrow{F} \mathcal{C} \text{ is an equivalence of categories}}$
b) Moreover, $X \mapsto (X)$ defines a functor $\mathcal{C} \xrightarrow{G} \mathcal{C}^{str}$ inverse to F

Our key objective now is to show $\mathcal{C}^{str}$ – strict $\otimes$-category.

- $\otimes$-product on objects of $\mathcal{C}^{str}$

  Given two sequences $S = (X_1, \ldots, X_k)$, $S' = (X'_1, \ldots, X'_l)$ define
  $$S \otimes S' = (X_1, \ldots, X_k, X'_1, \ldots, X'_l) = \underbrace{S * S'}_{\text{shortcut for concatenation}}$$
  If $k=0$ or $l=0$, then we just set $\emptyset \otimes S = S = S \otimes \emptyset$.

  Clear: $(S \otimes S') \otimes S'' = S \otimes (S' \otimes S'')$ $\forall S, S', S'' \in \mathrm{Ob}\,\mathcal{C}^{str}$.

- natural isomorphisms $\varphi_{S,S'}: F(S) \otimes F(S') \to F(S \otimes S')$ $\forall S, S' \in \mathrm{Ob}\,\mathcal{C}^{str}$.

  Define those by induction on the length of $S' = (X'_1, \ldots, X'_l)$
  - If $l=0 \Rightarrow S' = \emptyset \Rightarrow F(S') = I$. Then $\varphi_{S,\emptyset} := r_{F(X)}$
  - If $l=1 \Rightarrow S' = (X'_1) \Rightarrow F(S') = X'_1 = S'$. Then $\varphi_{S,X'_1} := id_{F(S) \otimes X'_1}$.
  - If $l>1$, then write $S' = S'' * (X'_l)$ with $S'' = (X'_1, \ldots, X'_{l-1})$ of smaller length and define $\varphi_{S,S'}$ through the following commutative diagram

$$\begin{tikzcd}
F(S)\otimes F(S') = F(S)\otimes(F(S'')\otimes X'_l) \arrow[r, "a^{-1}_{F(S),F(S''),X'_l}", "iso"'] \arrow[d, dashed, "\varphi_{S,S'}"] & (F(S)\otimes F(S''))\otimes X'_l \arrow[d, "\varphi_{S,S''}\otimes id_{X'_l}", "iso"'] \\
F(S\otimes S') \arrow[r, equal] & F(S\otimes S'')\otimes X'_l
\end{tikzcd}$$

- $\otimes$-product on morphisms of $\mathcal{C}^{str}$

  Given $f: S \to T$, $g: S' \to T'$ in $\mathcal{C}^{str}$ (i.e. morphisms $f: F(S) \to F(T)$, $g: F(S') \to F(T')$ in $\mathcal{C}$) we define $f \otimes g: S \otimes S' \to T \otimes T'$ through commut. diagram:

$$\begin{tikzcd}
F(S)\otimes F(S') \arrow[r, "\varphi_{S,S'}", "iso"'] \arrow[d, "f\otimes g"] & F(S\otimes S') \arrow[d, dashed, "f\otimes g := \varphi_{T,T'}\circ(\underbrace{f\otimes g}_{\text{in } \mathcal{C}})\circ\varphi^{-1}_{S,S'}"] \\
F(T)\otimes F(T') \arrow[r, "\varphi_{T,T'}"] & F(T\otimes T')
\end{tikzcd}$$

[Exercise

$\boxed{\text{Proposition 1: } \mathcal{C}^{str} \text{ with above } \otimes \text{ is a strict tensor category, tensor equivalent to } \mathcal{C}}$