Goal: introduce the notion of braidings on $\otimes$-categories, discuss the braid category $\mathcal{B}$.

Let $\mathcal{C}$ be a category and $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ - "$\otimes$-product" functor equipped with an associativity constraint a (but we don't yet assume anything about unit object!)

Def: A commutativity constraint c is a natural isomorphism $\otimes \xrightarrow{\simeq} \otimes\tau$

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, bend left, "\otimes\tau"] \arrow[r, bend right, "\otimes"'] \arrow[r, phantom, "\wr\Downarrow c"] & \mathcal{C} \end{tikzcd}$$

where $\tau : \mathcal{C}\times\mathcal{C} \xrightarrow{\simeq} \mathcal{C}\times\mathcal{C}$ - "flip" functor

Down-to-earth that means that one has isomorphisms

$$c_{X,Y} : X\otimes Y \xrightarrow{\simeq} Y\otimes X \qquad \forall X, Y \in \mathrm{Ob}\,\mathcal{C}$$

which are natural in the sense of commutativity of the following square:

$$\begin{tikzcd} X\otimes Y \arrow[r, "c_{X,Y}"] \arrow[d, "f\otimes g"'] & Y\otimes X \arrow[d, "g\otimes f"] \\ X'\otimes Y' \arrow[r, "c_{X',Y'}"] & Y'\otimes X' \end{tikzcd} \qquad \forall\, f : X \to X',\ g : Y \to Y'.$$

Def: The commutativity constraint c satisfies Hexagon Axiom if the following hexagon diagrams commute

$$\begin{tikzcd} & X\otimes(Y\otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & (Y\otimes Z)\otimes X \arrow[dr, "a_{Y,Z,X}"] & \\ (X\otimes Y)\otimes Z \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "c_{X,Y}\otimes id_Z"'] & & & Y\otimes(Z\otimes X) \\ & (Y\otimes X)\otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y\otimes(X\otimes Z) \arrow[ur, "id_Y\otimes c_{X,Z}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & X\otimes(Y\otimes Z) \arrow[r, "c^{-1}_{Y\otimes Z,X}"] & (Y\otimes Z)\otimes X \arrow[dr, "a_{Y,Z,X}"] & \\ (X\otimes Y)\otimes Z \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "c^{-1}_{Y,X}\otimes id_Z"'] & & & Y\otimes(Z\otimes X) \\ & (Y\otimes X)\otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y\otimes(X\otimes Z) \arrow[ur, "id_Y\otimes c^{-1}_{Z,X}"'] & \end{tikzcd}$$

The key notion for today is:

Def: A braiding on a tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is a commutativity constraint c satisfying the Hexagon Axiom. Then $(\mathcal{C}, \otimes, I, a, l, r, c)$ is called a braided tensor category.

Remark: If $\mathcal{C}$ is strict (so that $a_{\cdot,\cdot,\cdot}$ - identities) then the Hexagon axiom reads:

$$\boxed{c_{X,Y\otimes Z} = (id_Y\otimes c_{X,Z})(c_{X,Y}\otimes id_Z) \quad \text{and} \quad c_{X\otimes Y,Z} = (c_{X,Z}\otimes id_Y)(id_X\otimes c_{Y,Z})}$$

Lecture #41

While the compatibility b/w a & c is in the sense of Hexagon Axiom, we note that we didn't request any compatibility of c with l & r, because this becomes a property:

> **Lemma 1**: $l_X \circ c_{X,I} = r_X$, $r_X \circ c_{I,X} = l_X$, $c_{I,X} = c_{X,I}^{-1}$ for any $X \in \mathrm{Ob}\,\mathcal{C}$

The first two equalities correspond to the following commutative triangles:

$$\begin{tikzcd} X \times I \arrow[rr, "c_{X,I}"] \arrow[dr, "r_X"'] & & I \times X \arrow[dl, "l_X"] \\ & X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I \times X \arrow[rr, "c_{I,X}"] \arrow[dr, "l_X"'] & & X \times I \arrow[dl, "r_X"] \\ & X & \end{tikzcd}$$

a) Consider the following bigger diagram

$$\begin{tikzcd} (X\otimes I)\otimes Y \arrow[rr, "a"] \arrow[dr, "r\otimes id"] \arrow[dd, "c\otimes id"'] & & X\otimes(I\otimes Y) \arrow[rr, "c"] \arrow[dl, "id\otimes l"'] & & (I\otimes Y)\otimes X \arrow[dl, "l\otimes id"'] \arrow[dd, "a"] \\ & X\otimes Y \arrow[rr, "c"] & & Y\otimes X & \\ (I\otimes X)\otimes Y \arrow[rr, "a"'] \arrow[ur, "l\otimes id"] & & I\otimes(X\otimes Y) \arrow[rr, "id\otimes c"'] \arrow[ul, "l"'] & & I\otimes(Y\otimes X) \arrow[ul, "l"'] \end{tikzcd}$$

Exercise: Add all the indices in $a_{\cdot,\cdot,\cdot}$, $c_{\cdot\cdot}$, $l_\cdot$, $r_\cdot$

Let's now detect commutative subdiagrams inside:

1) outer hexagon – due to Hexagon axiom
2) top-right square – due to naturality of c
3) bottom-right square – due to naturality of l
4) top-left triangle – due to Triangle axiom
5) bottom-left triangle, right-middle triangle – due to [Lecture 36, Lemma 2]

Chasing the diagram, we can now deduce the commutativity of the left-middle triangle (to see it, compose both $r\otimes id : (X\otimes I)\otimes Y \to X\otimes Y$ and $(l\otimes id)(c\otimes id)$ with $X\otimes Y \xrightarrow{c} Y\otimes X \xrightarrow{l^{-1}_{Y\otimes X}} I\otimes(Y\otimes X)$)

Setting $Y = I$, this left-middle triangle implies the desired $l_X \circ c_{X,I} = r_X$.

b) Replacing c by c^{-1} in the above argument one gets $r_X \circ c_{I,X} = l_X$

c) Combining a) & b), the equality $c_{I,X} = c_{X,I}^{-1}$ is immediate

Lecture #41

Our next property can be viewed as a "categorical version of Yang-Baxter eq-n":

Proposition 1: For any objects X, Y, Z in a braided tensor category $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
 & (X\otimes Y)\otimes Z \arrow[dl, "c\otimes id"'] \arrow[dr, "a"] \arrow[dddddr, dashed, "c_{X\otimes Y, Z}"] & \\
(Y\otimes X)\otimes Z \arrow[d, "a"] \arrow[dddddr, dashed, "c_{Y\otimes X, Z}"] & & X\otimes(Y\otimes Z) \arrow[d, "id\otimes c"] \\
Y\otimes(X\otimes Z) \arrow[d, "id\otimes c"] & & X\otimes(Z\otimes Y) \arrow[d, "a^{-1}"] \\
Y\otimes(Z\otimes X) \arrow[d, "a^{-1}"] & & (X\otimes Z)\otimes Y \arrow[d, "c\otimes id"] \\
(Y\otimes Z)\otimes X \arrow[d, "c\otimes id"] & & (Z\otimes X)\otimes Y \arrow[d, "a"] \\
(Z\otimes Y)\otimes X \arrow[dr, "a"] & & Z\otimes(X\otimes Y) \arrow[dl, "id\otimes c"] \\
 & Z\otimes(Y\otimes X) &
\end{tikzcd}$$

▶ Divide the above 12 vertices into 2 hexagons by drawing punctured lines. The resulting two hexagons commute due to Hexagon Axiom. It thus remains to verify the commutativity of the square

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "c\otimes id"] \arrow[d, "c_{X\otimes Y, Z}"'] & (Y\otimes X)\otimes Z \arrow[d, "c_{Y\otimes X, Z}"] \\
Z\otimes(X\otimes Y) \arrow[r, "id\otimes c"'] & Z\otimes(Y\otimes X)
\end{tikzcd}$$

– but this is naturality of the braiding c w.r.t $X\otimes Y \xrightarrow{c} Y\otimes X$ and $Z \xrightarrow{id_Z} Z$ ■

Remark: a) If $\mathcal{C}$ - strict $\otimes$-category (so that all a...- identities), the above reads

$$(c_{Y,Z}\otimes id_X)(id_Y\otimes c_{X,Z})(c_{X,Y}\otimes id_Z) = (id_Z\otimes c_{X,Y})(c_{X,Z}\otimes id_Y)(id_X\otimes c_{Y,Z})$$

In particular, if $Y=Z=X$, then we get equality:

$$(c_{X,X}\otimes id_X)(id_X\otimes c_{X,X})(c_{X,X}\otimes id_X) = (id_X\otimes c_{X,X})(c_{X,X}\otimes id_X)(id_X\otimes c_{X,X})$$

b) Same will apply in $Vect_{\mathbb{k}}$, which is not strict, b/c associativity constraint there is easy.

Lecture # 41

We shall now discuss some examples of braided $\otimes$-categories, some of which are symmetric, i.e. $c_{Y,X}\circ c_{X,Y}=id_{X\otimes Y}$ $\forall X,Y\in Ob\,\mathcal{C}$.

Example 0: $\mathcal{C}=Vect_{\Bbbk}$ or $Vect_{\Bbbk}^{f.d}$ with $c=\tau$ - flip

A slight generalization of this is:

Example 1: Let H be a cocommutative bialgebra, so that $\Delta(a)=\Delta^{op}(a)$ $\forall a\in H$. Then H-Mod is a braided $\otimes$-category with a braiding $c=\tau$ - flip. In particular, this applies to $H=\Bbbk[G]$ or $H=\mathcal{U}(\mathfrak{g})$

The next important example features some basic patterns and is motivation for B discussed afterwards

Example 2a: Consider GL with $Ob(GL)=\mathbb{Z}_{\geq 0}$, $Hom_{GL}(k,l)=\begin{cases}\emptyset & \text{if } k\neq l\\ GL_l(\Bbbk) & \text{if } k=l>0\\ \{e\} & \text{if } k=l=0\end{cases}$ where the composition is the gp product.

This is a strict $\otimes$-category with tensor product defined by:

- on objects $n\otimes m:=n+m$
- on morphisms $\begin{matrix} n\\ \downarrow f\\ n\end{matrix}\otimes\begin{matrix} m\\ \downarrow g\\ m\end{matrix}:=\begin{matrix} n+m\\ \downarrow f\otimes g\\ n+m\end{matrix}$ with matrix of $f\otimes g=\left(\begin{array}{c|c} f & 0\\ \hline 0 & g\end{array}\right)$
- unit element $I=0$

In this case, we claim that GL is actually braided with

$c_{n,m}: n\otimes m \xrightarrow{\sim} m\otimes n$ given by matrix $\left(\begin{array}{c|c} 0 & I_m\\ \hline I_n & 0\end{array}\right)$ which precisely corresponds to swapping n & m vectors

Lemma 2: This $\{c_{n,m}\}$ defines a symmetric braiding on GL.

▷ • Functoriality follows from
$$\begin{pmatrix}0 & I_m\\ I_n & 0\end{pmatrix}\begin{pmatrix}f & 0\\ 0 & g\end{pmatrix}=\begin{pmatrix}0 & g\\ f & 0\end{pmatrix}=\begin{pmatrix}g & 0\\ 0 & f\end{pmatrix}\begin{pmatrix}0 & I_m\\ I_n & 0\end{pmatrix}$$

• Symmetry follows from
$$\begin{pmatrix}0 & I_m\\ I_n & 0\end{pmatrix}\begin{pmatrix}0 & I_n\\ I_m & 0\end{pmatrix}=I_{n+m}$$

• Finally, hexagon axiom follows easily from $c_{n,m\otimes k}=(id_m\otimes c_{n,k})(c_{n,m}\otimes id_k)$ ∎

Example 2b: Viewing S_n as a subgp of $GL_n(\Bbbk)$ one likewise obtains a braiding on the similar category $\mathcal{S}$ with $Ob(\mathcal{S})=\mathbb{Z}_{\geq 0}$ and $Hom_{\mathcal{S}}(k,l)=\begin{cases}\emptyset, & \text{if } k\neq l\\ S_\ell, & \text{if } k=l>0\\ \{e\}, & \text{if } k=l=0\end{cases}$

Lecture #41

The next topological example is very similar to the tangle category $\mathcal{T}$.

Example 3: Braid category $\mathcal{B}$

$Ob(\mathcal{B})$ = {all finite sequences of +'s} $\leftrightarrow \mathbb{Z}_{\geq 0}$

$Hom(\mathcal{B})$ = Braids on any number of strands / equivalence

Unit element $I \in \mathcal{B}$ is $0 \in \mathbb{Z}_{\geq 0}$ (corresponding to an empty sequence)

Composition in $\mathcal{B}$ - is the composition of braids

$\otimes$-product of objects $n \otimes m = n+m$

$\otimes$-product of morphisms is stacking left-to-right

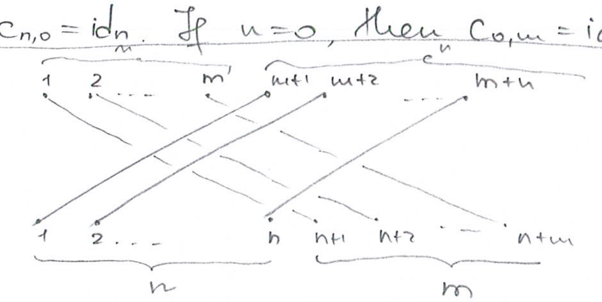
(e.g. $[X_+] \otimes [X_-]$ is the morphism $4 \longrightarrow 4$ given by )

Clear: The above makes $\mathcal{B}$ into a strict $\otimes$-category

We shall now endow $\mathcal{B}$ with a braiding. To do so, we need to specify a braid (up to isotopy equivalence) representing $c_{n,m}: n \otimes m \longrightarrow m \otimes n$.

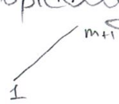
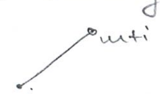
If $m=0$, then $c_{n,0} = id_n$. If $n=0$, then $c_{0,m} = id_m$. Finally, for $n, m \geq 1$ set:

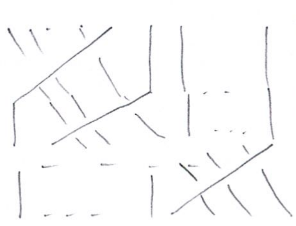
$c_{n,m}$:



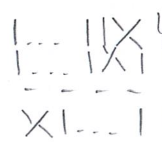
Algebraically, such a braid may be written as

$$\boxed{c_{n,m} = (\sigma_m \sigma_{m-1} \dots \sigma_1)(\sigma_{m+1} \sigma_m \dots \sigma_2) \dots (\sigma_{m+n-1} \dots \sigma_n)}$$

to see this isotopically deform above braid to let all the "crossing points" of the braid  be higher than those of other , i.e.



$\leftarrow$ slice it further into

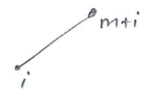
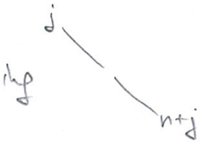


Proposition 2: This defines a braiding on $\mathcal{B}$

Lecture #41

► Proof of Prop 2

- Functoriality amounts to verifying $c_{n,m}\circ(f\otimes g)\sim(g\otimes f)\circ c_{n,m}$ for any two braids f, g on n, m strands

  Topologically: this is clear as all braids  are overcrossing 

  Hence, applying (R3) transformation multiple times, we can "pull" braids f, g along the braids in $c_{n,m}$ from bottom to top.

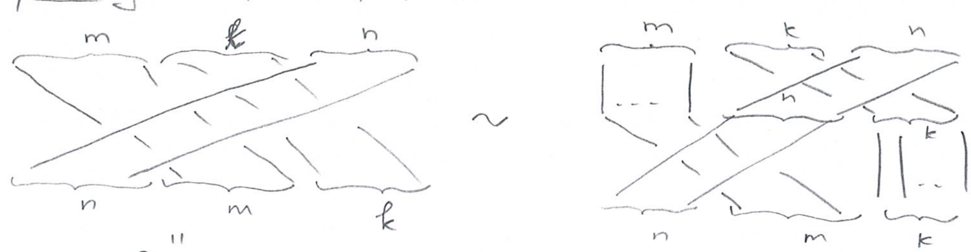
  Algebraically: For simplicity, let's just illustrate the smallest nontrivial case $n=m=2$, $f=\sigma_1=g \Rightarrow f\otimes g=\sigma_1\sigma_3=g\otimes f$. Then all we are going to use is the braid rel-n $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j=\sigma_j\sigma_i$ for $|i-j|>1$:

$$\begin{aligned}(g\otimes f)c_{2,2} &= \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2 = \sigma_3\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 = \\ &= \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3 = \sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3 = \\ &= \sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3 = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3 = c_{2,2}(f\otimes g)\end{aligned}$$

- Hexagon Axiom

  Since $\mathcal{B}$-strict, just need to check 2 equalities from the end of p. 1.

  1st equality: $X=n$, $Y=m$, $Z=k$



$= C_{X,Y\otimes Z}$ $\qquad$ $= (id_Y\otimes C_{X,Z})(C_{X,Y}\otimes id_Z)$

  2nd equality: -//-

$= C_{X\otimes Y,Z}$ $\qquad$ $= (C_{X,Z}\otimes id_Y)(id_X\otimes C_{Y,Z})$