Lecture # 42

Goal: Braided bialgebras, cobraided bialgebras, and the FRT construction.

Last time we discussed braided tensor categories and at the very end asked:

Q: Given a bialgebra H, which conditions on H are equivalent to H-Mod being braided?

To answer this, we start with the following key definition:

Def.: a) A bialgebra H is quasi-cocommutative if $\exists R \in H \otimes H$ satisfying:

$$\boxed{R\text{-invertible} \quad \text{and} \quad \Delta^{op}(x) = R\Delta(x)R^{-1} \quad \forall x \in H}$$

Such R is called a universal R-matrix

b) A quasi-cocommutative bialgebra H with universal R-matrix R is braided if:

$$\boxed{(\Delta \otimes id)(R) = R_{13}R_{23} \quad \text{and} \quad (id \otimes \Delta)(R) = R_{13}R_{12}}$$

Finally, if H had the antipode S (thus being a Hopf algebra), then we say that H is a quasi-cocommutative Hopf algebra, respectively braided Hopf alg, if the above holds.

Proposition 1: If $(H, \mu, \eta, \Delta, \varepsilon, R)$ is a braided bialgebra, then:

a) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

b) $(\varepsilon \otimes id)(R) = 1 = (id \otimes \varepsilon)(R)$

If furthermore H is a Hopf algebra with antipode S, then:

c) $(S \otimes id)(R) = R^{-1} = (id \otimes S^{-1})(R)$

d) $(S \otimes S)(R) = R$

▶ a) $R_{12}R_{13}R_{23} = R_{12}(\Delta \otimes id)(R) = (\Delta^{op} \otimes id)(R)R_{12} = (\tau \otimes id)\underbrace{(\Delta \otimes 1)(R)}_{R_{13}R_{23}} \cdot R_{12} = R_{23}R_{13}R_{12}$

b) $R \overset{b/c\ (\varepsilon \otimes id)\Delta = id}{=} (\varepsilon \otimes id \otimes id)(\Delta \otimes id)(R) = (\varepsilon \otimes id \otimes id)(R_{13}R_{23}) = (\varepsilon \otimes id)R \cdot R \underset{R\text{-invertible}}{\Longrightarrow} (\varepsilon \otimes id)(R) = 1$

Using $(id \otimes \varepsilon)\Delta = id$ instead one gets $(id \otimes \varepsilon)(R) = 1$

c) $\underbrace{(\mu \otimes id)(S \otimes id \otimes id)\underbrace{(\Delta \otimes id)(R)}_{R_{13}R_{23}}}_{= (S \otimes id)(R) \cdot R} \underset{b/c\ \mu(S \otimes id)\Delta = \varepsilon}{=} (\varepsilon \otimes id)(R) \overset{b)}{=} 1 \Rightarrow (S \otimes id)(R) = R^{-1}$.

The other equality is similar.

d) $(S \otimes S)(R) = (id \otimes S)(S \otimes id)R \overset{c)}{=} (id \otimes S)(id \otimes S^{-1})(R) = (id \otimes id)(R) = R$ ■

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Given any $V, W \in H\text{-Mod}$, where $(H, \mu, 1_H, \Delta, \varepsilon, R)$ – braided bialgebra, define

$$\boxed{\begin{array}{l} c^R_{V,W} : V\otimes W \xrightarrow{\sim} W\otimes V \\ \qquad\quad v\otimes w \longmapsto \tau_{V,W}(R(v\otimes w)) \end{array}}$$

Proposition 2: The above maps $c^R_{V,W}$ satisfy:

a) $c^R_{V,W}$ – isomorphism of H-modules

b) $\forall U, V, W \in H\text{-Mod}$:

$$c^R_{U\otimes V, W} = (c^R_{U,W} \otimes id_V)(id_U \otimes c^R_{V,W})$$

$$c^R_{U, V\otimes W} = (id_V \otimes c^R_{U,W})(c^R_{U,V} \otimes id_W)$$

c) $\forall U, V, W \in H\text{-Mod}$:

$$(c^R_{V,W} \otimes id_U)(id_V \otimes c^R_{U,W})(c^R_{U,V} \otimes id_W) = (id_W \otimes c^R_{U,V})(c^R_{U,W} \otimes id_V)(id_U \otimes c^R_{V,W})$$

▶ This is an immediate consequence of the defining relations of R (combined with τ).

a) $c^R_{V,W}(x(v\otimes w)) = \tau_{V,W} R\Delta(x)(v\otimes w) = \tau_{V,W}\Delta^{op}(x) R(v\otimes w) = \Delta(x)\tau_{V,W} R(v\otimes w)$

$$\Delta(x)\tau_{V,W} R(v\otimes w) = x(c^R_{V,W}(v\otimes w))$$

b) It will be useful to write one of the braiding cond-s on R as:

$$\sum_{i,j} s_i s_j \otimes t_j \otimes t_i = \sum_{i,(t_i)} s_i \otimes t_i' \otimes t_i'' \quad \text{where} \quad R = \sum_i s_i \otimes t_i \in H\otimes H.$$

Then, the 2^{nd} equality of b) follows:

$$(id_V \otimes c^R_{U,W})(c^R_{U,V} \otimes id_W)(u\otimes v\otimes w) = (id_V \otimes c^R_{U,W})(\textstyle\sum_i t_i v \otimes s_i u \otimes w) =$$

$$= \sum_{i,j} t_i v \otimes t_j w \otimes s_j s_i u \xlongequal[\text{equality}]{\text{above}} \sum_{i,(t_i)} t_i' v \otimes t_i'' w \otimes s_i u = \underbrace{\sum_i t_i(v\otimes w)\otimes s_i(u)}_{= c^R_{U,V\otimes W}(u\otimes v\otimes w)}$$

The 1^{st} equality of b) follows likewise

c) Follows from Prop 1a), which can be interpreted as:

$$\sum_{i,j,k} s_k s_j \otimes t_k s_i \otimes t_j t_i = \sum_{i,j,k} s_j s_i \otimes s_k t_i \otimes t_k t_j$$

Indeed:

$$\text{LHS}(u\otimes v\otimes w) = (c^R_{V,W}\otimes id_U)(id_V \otimes c^R_{U,W})(\textstyle\sum_i t_i v\otimes s_i u\otimes w) = (c^R_{V,W}\otimes id_U)(\sum_{i,j} t_i v\otimes t_j w\otimes s_j s_i u)$$

$$\xlongequal[\text{equality}]{\text{above}} \sum_{i,j,k} t_k t_j w \otimes s_k t_i v \otimes s_j s_i u$$

$$\text{RHS}(u\otimes v\otimes w) = (id_W \otimes c^R_{U,V})(c^R_{U,W}\otimes id_V)(\textstyle\sum_i u\otimes t_i w\otimes s_i v) = (id_W\otimes c^R_{U,V})(\sum_{i,j} t_j t_i w\otimes s_j u\otimes s_i v)$$

$$= \sum_{i,j,k} t_j t_i w \otimes t_k s_i v \otimes s_k s_j u$$

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Def: Given a vector space V, an ~~endomorphism~~ invertible endomorphism $c \in \mathrm{Aut}(V\otimes V)$ is called an *R-matrix* if it satisfies the Yang–Baxter equation: $c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23}$ (viewed as equality in $\mathrm{End}(V\otimes V\otimes V)$ with $c_{12} = c\otimes \mathrm{id}_V$, $c_{23} = \mathrm{id}_V \otimes c$)

Taking $U = V = W$ in Proposition 2c), we thus get:

Corollary 1: If $(H, \mu, \eta, \Delta, \varepsilon, R)$ - braided bialgebra, then for any $V \in H\text{-Mod}$, $c^R_{V,V} \in \mathrm{End}(V\otimes V)$ is an R-matrix (in above sense)

Now we can finally answer the question from the very beginning:

Proposition 3: H-Mod is braided $\Leftrightarrow$ H is braided

$\boxed{\Leftarrow}$

The braiding on the category H-Mod is precisely given by $\{c^R_{V,W} \mid V, W \in H\text{-Mod}\}$

The conditions of a braiding on a category then follow from Prop. 2

$\boxed{\Rightarrow}$

Assume we have a braiding on the category H-Mod. First, we produce R:

$$\boxed{R := \tau_{H,H}\,(c_{H,H}(1\otimes 1))} \qquad \text{where} \quad c_{H,H} : H\otimes H \longrightarrow H\otimes H$$

Then: 1) naturality of $c_{\cdot,\cdot}$ will imply R - universal R-matrix
2) hexagon axiom will imply the braiding equalities

But first we note that the above construction is indeed "inverse" to $\Leftarrow$:

for any $V, W \in H\text{-Mod}$, and any $v \in V, w \in W$, consider morphisms $H \xrightarrow{\varphi_v} V$, $H \xrightarrow{\varphi_w} W$ s.t. $\varphi_v(h) = hv$, $\varphi_w(h) = hw$. Then naturality of braiding implies:

$$\begin{tikzcd}
H\otimes H \arrow[r, "c_{H,H}"] \arrow[d, "\varphi_v\otimes\varphi_w"'] & H\otimes H \arrow[d, "\varphi_w\otimes\varphi_v"] \\
V\otimes W \arrow[r, "c_{V,W}"'] & W\otimes V
\end{tikzcd}$$

Tracing the images of $1\otimes 1 \in H\otimes H$, we get: $c_{V,W}(v\otimes w) = \tau_{V,W}(R(v\otimes w)) = c^R_{V,W}(v\otimes w)$

Back to 1) from above. Note $c_{H,H}$ is an H-module morphism $\Rightarrow x\circ c_{H,H} = c_{H,H}\circ x$

$\forall x \in H \Rightarrow \Delta(x)\underbrace{(c_{H,H}(1\otimes 1))}_{\tau_{H,H} R} = c_{H,H}(\Delta(x)(1\otimes 1)) \overset{\text{above}}{=\!=} \tau_{H,H}(R\Delta(x)) \Rightarrow \Delta^{op}(x)R = R\Delta(x)$

$\forall x \in H$

As per 2) - direct chase of diagram

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By the above result, a braiding on the category H-Mod is equivalent to a braiding on the bialgebra H. There is a dual notion of cobraiding on H! To motivate the definition, let's view $H\otimes H$ as $\mathrm{Hom}_{\Bbbk}(\Bbbk, H\otimes H)$, so that $R\in H\otimes H$ from before is really a linear map $\Bbbk \longrightarrow H\otimes H$, $1 \longmapsto R$. Thus, following our Lectures 1-3, the dual object should be a linear map $H\otimes H \longrightarrow \Bbbk$.

Def: a) A bialgebra H is quasi-cocommutative if there is a linear form $\tau\in(H\otimes H)^*$ s.t.

$$\boxed{\begin{array}{l}\bullet\ \exists\,\bar\tau\in(H\otimes H)^* \text{ satisfying } \tau\star\bar\tau=\varepsilon=\bar\tau\star\tau\\ \bullet\ \mu^{op}=\tau\star\mu\star\bar\tau\end{array}}$$

← such τ is called a universal r-form

b) A quasi-cocommutative bialgebra H with universal r-form τ is cobraided if

$$\boxed{\tau(\mu\otimes id_H)=\tau_{13}\star\tau_{23} \quad \text{and} \quad \tau(id_H\otimes\mu)=\tau_{13}\star\tau_{12}}$$

where $\tau_{12}=\tau\otimes\varepsilon$, $\tau_{23}=\varepsilon\otimes\tau$, $\tau_{13}=(\varepsilon\otimes\tau)(\tau_{12}\otimes id)$

Here, $\star$ is the convolution (see Lecture 2). For example, the equality $\tau\star\bar\tau=\varepsilon$ means the following:

$$\forall\ x\otimes y\in H\otimes H \qquad \sum_{(x)(y)}\tau(x'\otimes y')\cdot\bar\tau(x''\otimes y'')=\varepsilon(x)\varepsilon(y)$$

Remark: Let's explain how these properties arise naturally by dualizing those for $R\in H\otimes H$ from p.1. To this end, we will draw diagrams and revert arrows.

(1) R-invertible means $\exists R^{-1}\in H\otimes H$ s.t. $R\cdot R^{-1}=1_{H\otimes H}=R^{-1}\cdot R$.

Viewing R as a linear map $\Bbbk \xrightarrow{R} H\otimes H$, $1\mapsto R$, R^{-1} as a linear map $\Bbbk \to H\otimes H$, $1\mapsto R^{-1}$

that means

$$\begin{tikzcd} \Bbbk=\Bbbk\otimes\Bbbk \arrow[r, "R\otimes R^{-1}"] \arrow[rrr, bend right, "1\mapsto 1\otimes 1"'] & (H\otimes H)\otimes(H\otimes H) \arrow[r, "id\otimes\tau\otimes id"] & (H\otimes H)\otimes(H\otimes H) \arrow[r, "\mu\otimes\mu"] & H\otimes H \end{tikzcd}$$

Reverting the arrows, we get (by swapping $\mu\rightsquigarrow\Delta$, $\tau\rightsquigarrow\tau$, $R\rightsquigarrow\tau$, $R^{-1}\rightsquigarrow\bar\tau$)

$$\begin{tikzcd} \Bbbk=\Bbbk\otimes\Bbbk & (H\otimes H)\otimes(H\otimes H) \arrow[l, "\tau\otimes\bar\tau"'] & (H\otimes H)\otimes(H\otimes H) \arrow[l, "id\otimes\tau\otimes id"'] & H\otimes H \arrow[l, "\Delta\otimes\Delta"'] \arrow[lll, bend left, "\varepsilon_H\cdot\varepsilon_H"] \end{tikzcd}$$

which precisely encodes $\tau\star\bar\tau=\varepsilon$.

Swapping the order $R\Leftrightarrow R^{-1}$ in the first diagram will yield a swapped $\tau\Leftrightarrow\bar\tau$ 2nd diagram encoding $\bar\tau\star\tau=\varepsilon$.

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(Continuation of Rmk)

(2) Let's now interpret the property $\Delta^{op} = R\Delta R^{-1}$ via diagrams:

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[rr, bend left, "\Delta^{op}"] \arrow[d, equal] & H\otimes H \arrow[r, "\tau"] & H\otimes H & \\
\Bbbk\otimes H\otimes\Bbbk \arrow[r, "R\otimes\Delta\otimes R^{-1}"'] & (H\otimes H)\otimes(H\otimes H)\otimes(H\otimes H) \arrow[r, "\tau_{34}\tau_{45}\tau_{23}"] & (H\otimes H\otimes H)\otimes(H\otimes H\otimes H) \arrow[u, "\mu^2\otimes\mu^2"']
\end{tikzcd}$$

Dualizing this diagram by reverting all arrows (swapping $\Delta \rightsquigarrow \mu$, $\mu \rightsquigarrow \Delta$) get:

$$\begin{tikzcd}
H \arrow[d, equal] & H\otimes H \arrow[l, "\mu"'] & H\otimes H \arrow[l, "\tau"'] \arrow[ll, bend right, "\mu^{op}"'] \arrow[d, "\Delta^2\otimes\Delta^2"] \\
\Bbbk\otimes H\otimes\Bbbk & (H\otimes H)\otimes(H\otimes H)\otimes(H\otimes H) \arrow[l, "\tau\otimes\mu\otimes\bar{\tau}"] & (H\otimes H\otimes H)\otimes(H\otimes H\otimes H) \arrow[l, "\tau_{23}\tau_{45}\tau_{34}"]
\end{tikzcd}$$

which precisely encodes $\mu^{op} = \tau \star \mu \star \bar{\tau}$

(3) Let's now interpret $(\Delta\otimes id)(R) = R_{13}R_{23}$ via:

$$\begin{tikzcd}
\Bbbk \arrow[r, "R"] \arrow[d, "R_{13}\otimes R_{23}"'] \arrow[drr, bend right=40, "R_{13}\cdot R_{23}"'] & H\otimes H \arrow[r, "\Delta\otimes id"] & H\otimes H\otimes H \arrow[d, equal] \\
(H\otimes H\otimes H)\otimes(H\otimes H\otimes H) \arrow[rr, "\mu_{H\otimes H\otimes H}"] & & H\otimes H\otimes H
\end{tikzcd}$$

Dualizing this diagram, we get

$$\begin{tikzcd}
\Bbbk & H\otimes H \arrow[l, "\tau"'] & H\otimes H\otimes H \arrow[l, "\mu\otimes id"'] \arrow[d, equal] \\
(H\otimes H\otimes H)\otimes(H\otimes H\otimes H) \arrow[u, "\tau_{13}\otimes\tau_{23}"] & & H\otimes H\otimes H \arrow[ll, "\Delta_{H\otimes H\otimes H}"'] \arrow[ull, bend left=40, "\tau_{13}\star\tau_{23}"]
\end{tikzcd}$$

Likewise, one can easily see that $\tau_{13}\star\tau_{12} = \tau(id_H\otimes\mu)$ is indeed dual to $(id\otimes\Delta)(R) = R_{13}R_{12}$.

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Similar to how a braiding on H was giving a braiding on the category H-Mod, we shall now see that a cobraiding on H gives a braiding on the category H-Comod.

The best way to see the construction is again by dualizing diagrams:

$$c^R_{V,W}:\ V\otimes W = \Bbbk\otimes V\otimes W \xrightarrow{R\otimes id} (H\otimes H)\otimes V\otimes W \xrightarrow{id\otimes\tau\otimes id} H\otimes V\otimes H\otimes W \xrightarrow{\mu_V\otimes\mu_W} V\otimes W \xrightarrow{\tau} W\otimes V$$

$\rightsquigarrow$ dualize

$$V\otimes W = \Bbbk\otimes V\otimes W \xleftarrow{\tau\otimes id} (H\otimes H)\otimes V\otimes W \xleftarrow{id\otimes\tau\otimes id} H\otimes V\otimes H\otimes W \xleftarrow{\Delta_V\otimes\Delta_W} V\otimes W \xleftarrow{\tau} W\otimes V$$

$\rightsquigarrow$ relabel $V\rightsquigarrow W$, $W\rightsquigarrow V$

$$\boxed{c^\tau_{V,W}:\ V\otimes W \xrightarrow{\tau} W\otimes V \xrightarrow{\Delta_W\otimes\Delta_V} (H\otimes W)\otimes(H\otimes V) \xrightarrow{\tau_{23}} H\otimes H\otimes W\otimes V \xrightarrow{\tau\otimes id} W\otimes V}$$

Using the Sweedler's notation $\Delta_V(v)=\sum_{(v)} v_H\otimes v_V$, $\Delta_W(w)=\sum_{(w)} w_H\otimes w_W$ (see Lecture 3), we can write an explicit algebraic f-la for $c^\tau_{V,W}$:

$$\boxed{c^\tau_{V,W}(v\otimes w) = \sum_{(v)(w)} \tau(w_H\otimes v_H)\cdot w_W\otimes v_V}$$

The following result is dual to Proposition 2

Proposition 4: The above maps $c^\tau_{V,W}$ satisfy:

a) $c^\tau_{V,W}$ – isomorphism of H-comodules

b) $\forall U,V,W\in H$-CoMod

$$c^\tau_{U\otimes V,W} = (c^\tau_{U,W}\otimes id_V)(id_U\otimes c^\tau_{V,W})$$

$$c^\tau_{U,V\otimes W} = (id_V\otimes c^\tau_{U,W})(c^\tau_{U,V}\otimes id_W)$$

c) $\forall U,V,W\in H$-CoMod

$$(c^\tau_{V,W}\otimes id_U)(id_V\otimes c^\tau_{U,W})(c^\tau_{U,V}\otimes id_W) = (id_W\otimes c^\tau_{U,V})(c^\tau_{U,W}\otimes id_V)(id_U\otimes c^\tau_{V,W})$$

! In other words, similarly to Prop 2 setup, the above result yields that any cobraided bialgebra H allows for a natural braiding on the $\otimes$-category of left H-comodules

[**Exercise**: Prove Prop. 4

Hint: The inverse map $\bar{c}^\tau_{V,W}: W\otimes V\to V\otimes W$ is given by $\bar{c}^\tau_{V,W}(w\otimes v)=\sum_{(v)(w)} \bar{\tau}(w_H\otimes v_H)\cdot v_V\otimes w_W$

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We end with the famous "FRT (= Faddeev - Reshetikhin - Takhtajan) construction". Following [Kassel, VIII.6] we split it into 2 theorems:

Theorem 1: Let V be a fin. dim. vector space and $c \in \mathrm{End}(V \otimes V)$. Then, there is a (unique up to isomorphism) bialgebra $A(c)$ with a linear map $\Delta_V : V \to A(c) \otimes V$ s.t.

1) Δ_V is a comodule map
2) $c : V \otimes V \circlearrowleft$ is a comodule morphism
3) $A(c)$ is "universal" in the sense that given any bialgebra A' coacting on V via $\Delta'_V : V \to A' \otimes V$ s.t. c - A'-comodule morphism $\exists!\ A(c) \xrightarrow{f} A'$ s.t.

$$\begin{tikzcd} V \arrow[r, "\Delta_V"] \arrow[rr, bend right, "\Delta'_V"'] & A(C) \otimes V \arrow[r, "f \otimes id"] & A' \otimes V \end{tikzcd}$$

Theorem 2: In the above setup if c is an R-matrix (i.e. invertible and satisfies Yang Baxter equation) then $\exists!$ form $r \in (A(c) \otimes A(c))^*$ making $A(c)$ into a cobraided bialgebra s.t.

$$C^{r}_{V,V} = c.$$

Let's sketch the proofs of these theorems

Proof of Thm 1

- First, we define $A(c)$ as an algebra by generators & relations. To this end, pick a basis $\{v_i\}_{i=1}^{N}$ of V and write c in the basis $\{v_i \otimes v_j\}_{i,j=1}^{N}$:

$$c(v_i \otimes v_j) = \sum_{m,n=1}^{N} C_{ij}^{mn}\, v_m \otimes v_n \quad , \text{ so that } (c_{ij}^{mn}) \text{- matrix coeff-s}$$

Define

$$\boxed{A(c) = \mathbb{k}\langle T_i^j \mid 1 \le i,j \le N \rangle / I(c)}$$

where $I(c)$ is the 2-sided ideal gen-d by

$$\boxed{A_{ij}^{mn} := \sum_{k,\ell=1}^{N} c_{ij}^{k\ell}\, T_k^m T_\ell^n - \sum_{k,\ell=1}^{N} C_{k\ell}^{mn}\, T_i^k T_j^\ell}$$

$\leftarrow$ these will come naturally when verifying 2)

- To define the bialgebra structure it's convenient to form an $N \times N$ matrix $T = (T_i^j)_{i,j=1}^{N}$ with T_i^j being the $(i,j)^{th}$-entry.

Lemma 1: $\exists!$ bialgebra structure on $A(c)$ with $\Delta(T) = T \otimes T$, $\varepsilon(T) = I_N$

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On the level of generators we get: $\boxed{\Delta(T_i^j) = \sum_k T_i^k \otimes T_k^j, \ \varepsilon(T_i^j) = \delta_{ij}}$

▷ The above f-las clearly define a bialg. structure on the free alg. gen-d by T_i^j.
Thus, we need to check $I(c)$ is a coideal:

1) $\varepsilon(A_{ij}^{mn}) = 0$ clearly

2) $\varepsilon(A_{ij}^{mn}) = \sum_{p,q} A_{ij}^{pq} \otimes T_p^m T_q^n + \sum_{p,q} T_i^p T_j^q \otimes A_{pq}^{mn}$

□

- Coaction of $A(c)$ on V is best written in matrix form

$$\boxed{\Delta_V \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = T \otimes \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}$$

i.e. $\boxed{\Delta_V(v_i) = \sum_{j=1}^{N} T_i^j \otimes v_j}$. The matrix/algebraic form easily imply it's coaction.

- The key part is to check c becomes a comodule morphism
Note, by definition, that: $\boxed{\Delta_{V\otimes V}(v_i \otimes v_j) = \sum_{k,l} T_i^k T_j^l \otimes v_k \otimes v_l}$

c - comodule morphism

$$\Updownarrow$$

$$\Delta_{V\otimes V}(c(v_i \otimes v_j)) - (id_{A(c)} \otimes c)(\Delta_{V\otimes V}(v_i \otimes v_j)) = 0 \quad \forall i,j$$

$$\Updownarrow$$

$$\sum_{m,n} \Big(\underbrace{\sum_{k,l} (c_{ij}^{kl} T_k^m T_l^n - c_{kl}^{mn} T_i^k T_j^l)}_{A_{ij}^{mn}} \Big) \otimes v_m \otimes v_n = 0 \quad \forall i,j$$

$$\Updownarrow$$

$A_{ij}^{mn} = 0 \quad \forall i,j,m,n$ ← which is by def-n of $A(c)$.

- Universal property is now immediate from the previous bullet!

□

Sketch of Proof of Theorem 2

▷ • Let's first deduce the values of τ on generators. To this end, we apply $c^{\tau}_{V,W}(v \otimes w) = \sum \tau(w_H \otimes v_H)\, w_W \otimes v_V$ in the context $W = V$, $H = A(c)$, $v = v_j$, $w = v_i$

$\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad c(v \otimes w)$

$$\left.\begin{aligned} \Rightarrow\ c(v_j \otimes v_i) &= \sum_{m,n} c_{ji}^{mn}\, v_m \otimes v_n \\ &\ \| \\ c^{\tau}_{V,V}(v_j \otimes v_i) &= \sum_{m,n} \tau(T_i^m, T_j^n)\, v_m \otimes v_n \end{aligned}\right\} \Rightarrow \boxed{\tau(T_i^m \otimes T_j^n) = c_{ji}^{mn}}$$

- Cobraiding properties recover τ on the entire algebra uniquely by knowing the values on generators. Thus, it suffices to show such τ exists

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(Continuation)

Exercise: Verify $\tau(T_p^q \otimes A_{ij}^{mn}) = 0 = \tau(A_{ij}^{mn} \otimes T_p^q)$ $\forall i,j,p,q$ is a consequence of YB on c where both $\tau(\ldots)$ are computed using cobraiding property and values on gens

Once such τ is now constructed, verifying its cobraiding is straightforward

Exercise: Verify $\tau \in (A(c) \otimes A(c))^*$ is a cobraiding

$\boxtimes$

We note that our construction above is very similar to that of $M_q(2)$ and its coaction on $\mathbb{k}^2$ from Lecture 6. In fact, $M_q(2)$ can be recovered that way:

Exercise: Let V be a 2-dim v.space with a basis $\{v_0, v_1\}$, then $V \otimes V$ is 4-dim with a basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$. Recall that

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1}-q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}(V \otimes V)$$

is a sol-n of YB eq-n (which was obtained in Lectures 14-15 by viewing V as $L(1,+) \in U_q(\mathfrak{sl}_2)$-mod, so that c is actually $U_q(\mathfrak{sl}_2)$-mod morphism)

Then: $A(c) \simeq M_{q^{-1}}(2)$ (compare to [Homework 6, Problem 2])