HOMEWORK 2 (DUE FEBRUARY 20)

1. Let $\phi\colon M^+_\lambda \to M^+_\mu$ be a nonzero homomorphism of Verma modules over a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ (with an abelian $\mathfrak{g}_0$). Prove that ϕ is injective.

Hint: You may wish to use the PBW theorem (stated in the form $\mathrm{gr}\, U(\mathfrak{n}_-) \simeq S(\mathfrak{n}_-)$).

2. Recall the notion of a <u>character</u> $\mathrm{ch}_V(q,x) := \sum_{d\in\mathbb{C}} q^{-d}\, \mathrm{Tr}_{V[d]}(e^x)$ for any $V \in \mathcal{O}^+$, where q is a formal variable and $x \in \mathfrak{h} = \mathfrak{g}_0$. Prove the formula

$$\mathrm{ch}_{M^+_\lambda}(q,x) = \frac{e^{\lambda(x)}}{\prod_{k>0} \det_{\mathfrak{g}_{-k}}(1 - q^k e^{\mathrm{ad}(x)})}$$

where the $\mathbb{C}$-grading on the Verma module M^+_λ is such that $\deg(v^+_\lambda) = 0$.

Hint: Use the PBW theorem, together with the equality $\sum_{n\geq 0} q^n \mathrm{Tr}_{S^n V}(S^n A) = \frac{1}{\det(1-qA)}$ for any endomorphism A of a finite-dimensional vector space V.

3. This problem is aimed at verification of two simple statements made in Lecture 6.

(a) Verify that for all $\mathbb{Z}$-graded Lie algebras discussed in Lectures 1–2, the map ω specified in Lecture 6 is indeed an involutive automorphism such that $\omega(\mathfrak{g}_k) = \mathfrak{g}_{-k}\ \forall\, k$ and $\omega|_{\mathfrak{g}_0} = -\mathrm{Id}$.

(b) Verify that the definition of the semidirect product $\mathfrak{g} \ltimes \mathfrak{a}$ indeed endows the vector space $\mathfrak{g} \oplus \mathfrak{a}$ with a Lie algebra structure.

4. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ with a <u>real structure</u> $\dagger$. Define

$$\mathfrak{g}_{\mathbb{R}} := \left\{ a \in \mathfrak{g} \,\middle|\, a^\dagger = -a \right\}$$

It is usually called the <u>real form</u> of $\mathfrak{g}$, due to the following result:

(a) Prove that $\mathfrak{g}_{\mathbb{R}}$ is a Lie algebra over $\mathbb{R}$ and $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$ as Lie algebras over $\mathbb{C}$.

Assume now that $\mathfrak{g}$ is $\mathbb{Z}$-graded with $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$, and $\mathfrak{g}_k^\dagger = \mathfrak{g}_{-k}$ for all k.

(b) Verify that $\overline{(M^-_{-\lambda})^{-\dagger}} \simeq M^+_\lambda$ as modules over $\mathfrak{g}$ iff λ is <u>real</u> (that is, $\lambda \in \mathfrak{g}^*_{0,\mathbb{R}}$).

(c) Prove that M^+_λ has a nonzero Hermitian form iff λ is real.

5. Let $\lambda, \mu \in \mathbb{C}$ and $\mathbf{i} := \sqrt{-1}$. Recall linear operators $\{\widetilde{L}_n\}_{n\in\mathbb{Z}}$ acting on F_μ from Lecture 7:

$$\widetilde{L}_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{n+j} + \mathbf{i}\lambda n a_n \ \text{ if } n \neq 0\,, \qquad \widetilde{L}_0 = \frac{\lambda^2+\mu^2}{2} + \sum_{j>0} a_{-j}a_j\,.$$

(a) Verify the following equality in $\mathrm{End}(F_\mu)$: $[\widetilde{L}_n, a_m] = -m a_{n+m} + \mathbf{i}\lambda m^2 \delta_{m,-n}\mathrm{Id}\ \forall\, m,n \in \mathbb{Z}$.

(b) Show that $\widetilde{L}_n$ define an action of Vir on F_μ with the central charge $c = 1 + 12\lambda^2$, i.e.

$$[\widetilde{L}_n, \widetilde{L}_m] = (n-m)\widetilde{L}_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}(1+12\lambda^2) \qquad \forall\, m,n \in \mathbb{Z}.$$

Note: This proves Proposition 2 of Lecture 7.

6. Let $\delta \in \{0, 1/2\}$. Recall the algebra C_δ (generated by the fermions $\{\psi_j\}_{j\in\delta+\mathbb{Z}}$) acting on the vector space V_δ (polynomials in anticommuting variables $\{\xi_j\}_{j\in\delta+\mathbb{Z}_{\geq 0}}$) from Lecture 8. Define the normal ordering $:\psi_i\psi_j: = \begin{cases} \psi_i\psi_j & \text{if } i \leq j \\ -\psi_j\psi_i & \text{if } i > j \end{cases}$ and recall $L_n \in \mathrm{End}(V_\delta)$ defined via

$$L_n = \delta_{n,0}\frac{1-2\delta}{16} + \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j :\psi_{-j}\psi_{n+j}: \qquad \forall\, n \in \mathbb{Z}\,.$$

(a) Verify the following equality in $\mathrm{End}(V_\delta)$: $[\psi_m, L_n] = \left(m + \frac{n}{2}\right)\psi_{m+n}$ $\forall\, m \in \delta+\mathbb{Z}, n \in \mathbb{Z}$.

(b) Show that L_n define an action of Vir on V_δ with the central charge $c = \frac{1}{2}$, i.e.

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{24} \qquad \forall\, m, n \in \mathbb{Z}.$$

Note: This proves Proposition 1 of Lecture 8.

For any algebra $\mathcal{C}$, we can split any quantum field $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$, with $A_n \in \mathcal{C}$, into:

$$A(z) = A_+(z) + A_-(z) \quad \text{with} \quad A_+(z) = \sum_{n\leq -1} A_n z^{-n-1},\ A_-(z) = \sum_{n\geq 0} A_n z^{-n-1}.$$

Given $A(z), B(z) \in \mathcal{C}[[z, z^{-1}]]$, define $:A(z)B(w): = A_+(z)B(w) + B(w)A_-(z)$.

7. Let $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1},\ T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ be the quantum fields with coefficients in the universal enveloping of the Heisenberg ($\mathcal{A}$) and Virasoro (Vir) algebras, respectively.

(a) Compute the difference $a(z)a(w) - :a(z)a(w):$ on the Fock representation F_μ. Present the corresponding power series by rational functions (depending only on $z - w$).

(b) Evaluate the difference $T(z)a(w) - :T(z)a(w):$ on the Fock representation F_μ, viewed as a $\mathrm{Vir} \ltimes \mathcal{A}$-module. Present the answer as a linear combination of $a(w)$ and its derivatives with coefficients being rational functions in $z - w$.

(c) Evaluate the difference $T(z)T(w) - :T(z)T(w):$ on the highest weight Vir representation with central charge c. Present the answer as a linear combination of $T(w)$ and its derivatives with coefficients being rational functions in $z - w$.

Recall the delta-function from Lecture 8:

$$\delta(w-z) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n = \frac{1}{z-w} + \frac{1}{w-z} \quad \text{with} \quad \frac{1}{z-w} = \sum_{n\geq 0} z^{-n-1}w^n.$$

8. Express $[a(z), a(w)],\ [T(z), a(w)],\ [T(z), T(w)]$ via $a(z), T(z), \delta(w-z)$ and its derivatives.

9. Let F_0 be the Fock module of $\mathcal{A}$, $1 \in F_0$ denote the highest weight vector, $1 \in F_0^*$ denote the lowest weight vector of the dual representation, and $a(z)$ be as in Problem 7. Prove:

$$\left\langle 1^*, a(z_1)\cdots a(z_{2n})1 \right\rangle = \sum_{\{\sigma\in S_{2n}\,:\, \sigma^2=1,\, \sigma(i)\neq i\ \forall i\}} \prod_{i<\sigma(i)} \frac{1}{(z_i - z_{\sigma(i)})^2}\,.$$