HOMEWORK 3 (DUE MARCH 6)

1. (a) Prove that $\Lambda^k V$ and $S^k V$ are irreducible representations of $\mathfrak{gl}_\infty$ for any $k \in \mathbb{N}$.

Hint: You may wish to use the classical counterpart (for finite-dimensional V).

(b) Verify that the action of $\mathfrak{gl}_\infty$ on $\Lambda^{\frac{\infty}{2},m}V$ constructed in the class is indeed an action.

Note: This proves Proposition 1 of Lecture 9.

(c) Verify the equality $(Aw_1, w_2) = (w_1, A^t w_2)$ for any $A \in \mathfrak{gl}_\infty$ and $w_1, w_2 \in \Lambda^{\frac{\infty}{2},m}V$ w.r.t. the Hermitian form defined in class (i.e. semi-infinite wedges form an orthonormal basis).

Note: This completes the proof of Proposition 3 from Lecture 9.

2. Recall the linear map $\hat{\rho}\colon \bar{\mathfrak{a}}_\infty \to \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V)$ from Lecture 10. Following our notations, let us represent $A \in \bar{\mathfrak{a}}_\infty$ as a 2×2 block matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ w.r.t. $\{v_i\}_{i\leq 0} \cup \{v_i\}_{i>0}$. Verify the explicit formula for $\alpha(A,B) := [\hat{\rho}(A), \hat{\rho}(B)] - \hat{\rho}([A,B]) \in \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V)$:

$$\alpha(A,B) = \mathrm{Tr}(B_{21}A_{12} - A_{21}B_{12}) \cdot \mathrm{Id}$$

Note: This proves Proposition 1 of Lecture 10.

3. For $\gamma, \beta \in \mathbb{C}$, recall the Lie algebra embedding $\overline{\varphi}_{\gamma,\beta}\colon W \hookrightarrow \bar{\mathfrak{a}}_\infty$ constructed in Lecture 10.

(a) Verify the following formula (with α as in Problem 2):

$$\alpha\left(\overline{\varphi}_{\gamma,\beta}(L_n), \overline{\varphi}_{\gamma,\beta}(L_m)\right) = \delta_{n,-m}\left(\frac{n^3-n}{12}c_\beta + 2nh_{\gamma,\beta}\right)$$

where $c_\beta = -12\beta^2 + 12\beta - 2$ and $h_{\gamma,\beta} = \frac{\gamma(\gamma+2\beta-1)}{2}$.

(b) According to part (a), we get a Lie algebra embedding (see [Lecture 10, Proposition 5])

$$\varphi_{\gamma,\beta}\colon \mathrm{Vir} \hookrightarrow \mathfrak{a}_\infty \quad \text{given by} \quad C \mapsto c_\beta K,\ L_n \mapsto \overline{\varphi}_{\gamma,\beta}(L_n) + \delta_{n,0}h_{\gamma,\beta}K.$$

Hence, there is a natural action of Vir on $\Lambda^{\frac{\infty}{2},m}V$ (depending on $\gamma, \beta \in \mathbb{C}$). Verify that $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \cdots \in \Lambda^{\frac{\infty}{2},m}V$ is a Vir highest weight vector of the highest weight

$$\left(\frac{(\gamma-m)(\gamma+2\beta-m-1)}{2}, -12\beta^2 + 12\beta - 2\right).$$

4. Verify the second formula from Theorem 1 of Lecture 11:

$$\Gamma^*(u) = u^{-m}z^{-1}\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right)$$

5. Let 1 (resp. ψ_0) be the highest weight vector of the bosonic space $\mathcal{B}^{(0)}$ (resp. fermionic space $\mathcal{F}^{(0)}$) and $\langle \cdot, \cdot \rangle$ be the contravariant form on that space.

(a) Compute the inner product $\langle 1, \Gamma(u_1) \cdots \Gamma(u_n) \Gamma^*(v_1) \cdots \Gamma^*(v_n) 1 \rangle$ by using the explicit "vertex operator" formulas for $\Gamma(u)$ and $\Gamma^*(u)$.

(b) Evaluate an analogous inner product $\langle \psi_0, X(u_1) \cdots X(u_n) X^*(v_1) \cdots X^*(v_n) \psi_0 \rangle$.

(c) Equating the results of parts (a) and (b), deduce the following identity:

$$\frac{\prod_{1 \le i < j \le n} (u_i - u_j) \cdot \prod_{1 \le i < j \le n} (v_i - v_j)}{\prod_{i,j=1}^{n} (u_i - v_j)} = (-1)^{\frac{n(n-1)}{2}} \det \left(\frac{1}{u_i - v_j} \right)_{i,j=1}^{n}$$

(d) Give an elementary proof of the identity from part (c).

6. Let d be the degree operator on the Fock space $\mathcal{B}^{(0)} = F_0$ (so that d multiplies each homogeneous element by its degree, where $\deg(x_i) = i$). Recall the operator $\Gamma(u, v)$ acting on $\mathcal{B}^{(0)}$ from Lecture 12:

$$\Gamma(u, v) = \exp \left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j} \right) \cdot \exp \left(- \sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j \right)$$

Prove the following equality of formal series:

$$\mathrm{Tr}_{\mathcal{B}^{(0)}} \left(\Gamma(u, v) q^d \right) = \prod_{n \ge 1} \frac{1 - q^n}{(1 - q^n u/v)(1 - q^n v/u)}$$

Hint: Compute first the trace of the operator $e^{\alpha x} e^{\beta \partial} q^{\gamma x \partial}$ acting on the space $\mathbb{C}[x]$.

7. Define $M(\infty) = \mathrm{Id} + \mathfrak{gl}_\infty$ and let $\mathrm{GL}(\infty) \subset M(\infty)$ be the subset of invertible elements.

(a) Show that the matrix multiplication makes $M(\infty)$ into a monoid and $\mathrm{GL}(\infty)$ into a group.

(b) Verify that the formula

$$A(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots) = (Av_{i_0}) \wedge (Av_{i_1}) \wedge (Av_{i_2}) \wedge \cdots$$

defines an action of the monoid $M(\infty)$ and a group $\mathrm{GL}(\infty)$ on $\mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2}, m} V$.

8. For $\tau \in \mathcal{F}^{(0)} \backslash \{0\}$, show that $\tau \in \Omega$ iff $S(\tau \otimes \tau) = 0$.

Hint: Deduce this result, stated as Theorem 3 of Lecture 12, from its finite-dimensional counterpart (or just apply similar arguments), see Theorem 2 from Lecture 12.