## HOMEWORK 5 (DUE APRIL 10)

1. Let $\mathfrak{g}$ be a simple Lie algebra with an invariant non-degenerate pairing $(\cdot,\cdot)$. Prove that the Casimir element (with respect to $(\cdot,\cdot)$) acts on the Verma $\mathfrak{g}$-module $M_\lambda$ as $(\lambda,\lambda+2\rho)\mathrm{Id}_{\mathrm{M}_\lambda}$, where $\rho$ is the half-sum of all positive roots of $\mathfrak{g}$.

2. Does any highest weight $\widehat{\mathfrak{g}}$-module admit a $\widetilde{\mathfrak{g}}$-action extending that of $\widehat{\mathfrak{g}}$?

3. Verify that if an admissible $\widehat{\mathfrak{g}}$-module $M$ of non-critical level is unitary as a $\widehat{\mathfrak{g}}$-module, then it is also unitary as a Vir-module, where Vir acts via the Sugawara construction (the corresponding antilinear anti-involutions $\dagger$ on $\widehat{\mathfrak{g}}$ and Vir were introduced in Lecture 7).

4. Verify that if $\mathfrak{g}$ is simple, $B'$ is an orthonormal basis of $\mathfrak{g}$ with respect to the standard invariant form $(\cdot,\cdot)$, $M$ is an admissible $\widehat{\mathfrak{g}}$-module of the critical level $k=-h^\vee$, then operators $T_n=\sum_{a\in B'}^{m\in\mathbb{Z}} :a_m a_{n-m}:$ $(n\in\mathbb{Z})$ commute with $\widehat{\mathfrak{g}}$-action.

5. Prove directly that the positive part $\mathfrak{n}_+$ of the Lie algebras $\mathfrak{sl}_3,\mathfrak{sp}_4$ is generated by $e_1,e_2$ subject to the corresponding two Serre relations.

6. (a) Compute explicitly the generalized Cartan matrices and the corresponding Dynkin diagrams for $\widehat{\mathfrak{g}}$ with $\mathfrak{g}$ being a classical simple finite dimensional Lie algebra (series $ABCD$).

(b) Compute explicitly the generalized Cartan matrices and the corresponding Dynkin diagrams for $\widehat{\mathfrak{g}}$ with $\mathfrak{g}$ being an exceptional simple finite dimensional Lie algebra (types $EFG$).

7. Let $A=(a_{ij})_{i,j=1}^n\in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Let $\widetilde{\mathfrak{g}}(A)$ be the Lie algebra of Lecture 20, and $\widetilde{\mathfrak{h}},\widetilde{\mathfrak{n}}_+,\widetilde{\mathfrak{n}}_-$ be the Lie subalgebras of $\widetilde{\mathfrak{g}}(A)$ generated by $\{h_i\}_{i=1}^n,\{e_i\}_{i=1}^n,\{f_i\}_{i=1}^n$, respectively.

(a) Let $\widetilde{\mathfrak{n}}'_+$ be the free Lie algebra generated by $\{e_i\}_{i=1}^n$. Show that the universal enveloping $U(\widetilde{\mathfrak{n}}'_+)$ is a free associative algebra in $\{e_i\}_{i=1}^n$.

(b) Let $\widetilde{\mathfrak{h}}'$ be an abelian Lie algebra with a basis $\{h_i\}_{i=1}^n$. Construct an action of $\widetilde{\mathfrak{h}}'$ on $\widetilde{\mathfrak{n}}'_+$ via derivations, so that $h_i(e_j)=a_{ij}e_j$.

(c) Construct an action of $\widetilde{\mathfrak{g}}(A)$ on $U(\widetilde{\mathfrak{h}}'\ltimes\widetilde{\mathfrak{n}}'_+)$.

(d) Deduce the Lie algebra isomorphisms $\widetilde{\mathfrak{h}}\simeq\widetilde{\mathfrak{h}}'$ and $\widetilde{\mathfrak{n}}_+\simeq\widetilde{\mathfrak{n}}'_+$.

(e) Show that the assignment $e_i\mapsto f_i, f_i\mapsto e_i, h_i\mapsto -h_i$ $(1\le i\le n)$ gives rise to a Lie algebra automorphism of $\widetilde{\mathfrak{g}}(A)$. Deduce that $\widetilde{\mathfrak{n}}_-$ is isomorphic to the free Lie algebra in $\{f_i\}_{i=1}^n$.

8. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and $L\mathfrak{g}=\mathfrak{g}[t,t^{-1}]$. For a $\mathfrak{g}$-representation $V$ and $z\in\mathbb{C}^\times$, define an *evaluation representation* $V(z)$ (with the underlying vector space $V$) of $L\mathfrak{g}$ as a composition of $L\mathfrak{g}\to\mathfrak{g}$ given by $at^n\mapsto z^n\cdot a$ $(a\in\mathfrak{g},n\in\mathbb{Z})$ and $\mathfrak{g}\to\mathrm{End}(V)$.

(a) Let $V_1,\ldots,V_n$ be irreducible nontrivial $\mathfrak{g}$-representations. Show that the $L\mathfrak{g}$-representation $V_1(z_1)\otimes\cdots\otimes V_n(z_n)$ is irreducible if and only if $z_i\neq z_j$ for $i\neq j$.

(b) When are two such irreducible representations isomorphic?

(c) Show that any irreducible finite dimensional $L\mathfrak{g}$-representation has the form as in (a).

*Hint: For an irreducible finite dimensional $L\mathfrak{g}$-module $V$, show that $L\mathfrak{g}$-action on $V$ factors through the action of the finite dimensional Lie algebra $\mathfrak{g} \otimes (\mathbb{C}[t, t^{-1}]/I)$, where $I \subset \mathbb{C}[t, t^{-1}]$ is an ideal of finite codimension. Apply the primary decomposition of $I$ and Lie's theorem.*