HOMEWORK 6 (DUE APRIL 29)

1. Recall the inner product $\langle\cdot,\cdot\rangle\colon P\times P\to\mathbb{C}$ from Lecture 23 defined by:

$$\langle\varphi+\alpha,\psi+\beta\rangle=\varphi(h_\beta)+\psi(h_\alpha)+(h_\alpha,h_\beta)$$

for any $\varphi,\psi\in\mathfrak{h}^*$ and $\alpha,\beta\in F$, whereas $P=\mathfrak{h}^*\oplus F$. Identifying $P\simeq\mathfrak{h}^*_{\mathrm{ext}}$ (as in Lecture 22), verify that the induced pairing $(\cdot,\cdot)\colon\mathfrak{h}_{\mathrm{ext}}\times\mathfrak{h}_{\mathrm{ext}}\to\mathbb{C}$ is given by

$$(h_{\alpha_i},h_{\alpha_j})=d_i^{-1}a_{ij},\quad (D_i,h_{\alpha_j})=(h_{\alpha_j},D_i)=\delta_{ij},\quad (D_i,D_j)=0.$$

Use this to derive a symmetric invariant non-degenerate pairing on $\mathfrak{g}_{\mathrm{ext}}(A)$.

2. Recall the simple reflections $r_i\in\operatorname{End}P$ from Lecture 25, defined via:

$$r_i(\chi)=\chi-\chi(h_i)\alpha_i.$$

Verify that $r_i^2=\mathrm{Id}$ and also that they preserve the pairing from Problem 1:

$$\langle r_i(x),r_i(y)\rangle=\langle x,y\rangle\qquad\forall\, x,y\in P.$$

3. Let M^+_λ (resp. M^-_λ) be the highest weight (resp. lowest weight) Verma module over a finite dimensional simple Lie algebra $\mathfrak{g}$. Let V be any $\mathfrak{h}$-diagonalizable module over $\mathfrak{g}$ ($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$). Establish a vector space isomorphism:

$$\operatorname{Hom}_{\mathfrak{g}}(M^+_\lambda\otimes M^-_\mu,V)\simeq V[\lambda+\mu].$$

4. Show that any short exact sequence $0\to L_\mu\to V\to L_\nu\to 0$ ($\mu,\nu\in P_+$) of integrable modules over a Kac-Moody algebra $\mathfrak{g}(A)$ is split. Deduce that every integrable module V of finite length in the category $\mathcal{O}$ over $\mathfrak{g}(A)$ is a direct sum of irreducible modules L_λ ($\lambda\in P_+$).

5. Show that Verma module M_λ over an extended Kac-Moody algebra $\mathfrak{g}_{\mathrm{ext}}(A)$ is irreducible for generic $\lambda\in P$. Specify a countable collection of hyperplanes outside of which it is true.

6. Let V be a module over an extended Kac-Moody algebra $\mathfrak{g}_{\mathrm{ext}}(A)$ from the category $\mathcal{O}$. Show that for a generic $\lambda\in P$ (more precisely, for λ outside of a countable collection of hyperplanes) the module $M_\lambda\otimes V$ is semisimple, and describe its decomposition into irreducibles.

7. Let $\mathfrak{g}$ be a simple Lie algebra with the Chevalley generators $\{e_i,f_i,h_i\}_{i=1}^r$. Set

$$x[n]:=x\cdot t^n\in L\mathfrak{g}=\mathfrak{g}[t,t^{-1}]\qquad\forall\, x\in\mathfrak{g}.$$

Find the defining relations between the elements $\big\{e_i[n],f_i[n],h_i[n]\big\}_{1\le i\le r}^{n\in\mathbb{Z}}$ generating $L\mathfrak{g}$.

8. **Vertex Operator Construction for $\widehat{\mathfrak{sl}}_2$**

Let $\{a_i\}_{i\in\mathbb{Z}}$ be the standard generators of the Heisenberg algebra $\mathcal{A}$. Let F_μ be the Fock representation over $\mathcal{A}$, and set $F:=\bigoplus_{m\in\mathbb{Z}}F_{\sqrt{2}m}$. Define vertex operators on F:

$$X_\pm(u):=\exp\left(\pm\sqrt{2}\sum_{n>0}\frac{a_{-n}}{n}u^n\right)\exp\left(\mp\sqrt{2}\sum_{n>0}\frac{a_n}{n}u^{-n}\right)S^{\pm1}u^{\pm\sqrt{2}a_0},$$

where S is the operator of shift $m\to m+1$ (compare to $\Gamma(u),\Gamma^*(u)$ of Lecture 11).

(a) Show that

$$X_a(u)X_b(v) = (u-v)^{2ab} :X_a(u)X_b(v): \text{ for any } a,b \in \{\pm\}$$

(by abuse of notations, we identify $\pm$ with ± 1 above). In particular,

$$X_a(u)X_b(v) = X_b(v)X_a(u)$$

in the sense that the matrix elements of both sides are series in u, v which converge (but in different regions!) to the same rational functions.

(b) Calculate $\langle 1, X_+(u_1)\cdots X_+(u_n)X_-(v_1)\cdots X_-(v_n)1\rangle$ for a highest weight vector $1 \in F_0$.

(c) Find the commutation relation between $X_\pm(u)$ and a_n.

(d) Show that the assignment

$$e(u) = \sum_{n\in\mathbb{Z}} e[n]u^{-n-1} \mapsto X_+(u),\ f(u) = \sum_{n\in\mathbb{Z}} f[n]u^{-n-1} \mapsto X_-(u),\ h[n] \mapsto \sqrt{2}a_n,\ K \mapsto \mathrm{Id}_F$$

defines an action of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ on F. Show that this is a level one highest weight representation of $\widehat{\mathfrak{sl}}_2$ with the highest weight 0 with respect to $\mathfrak{sl}_2$.

(e) Show that F is an irreducible $\widehat{\mathfrak{sl}}_2$-representation. Compute its character, i.e. $\mathrm{Tr}_F(e^{zh}q^{\mathsf{d}})$, where h is the Chevalley generator of $\mathfrak{sl}_2$ and d is the degree operator defined by the conditions $\mathsf{d}(1) = 0$ and $[\mathsf{d}, x[n]] = nx[n]$ for any $x \in \mathfrak{sl}_2$, $n \in \mathbb{Z}$.