Lecture #1

This course is devoted to the study of the structure and representation theory of some of the most important ∞-dim Lie algebras

Def 0: A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with a Lie bracket

$$[\,,\,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

s.t.

1) $[x,y]=-[y,x] \quad \forall x,y\in\mathfrak{g}$ ← skew-symmetry
2) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0 \quad \forall x,y,z\in\mathfrak{g}$ ← Jacobi rel.

We shall discuss this week the following key ∞-dim Lie algebras:

Heisenberg algebra, Virasoro algebra, affine Kac-Moody algebras

Unless stated otherwise, we usually assume the ground field to be $\Bbbk=\mathbb{C}$.

Def 1: The *Heisenberg algebra* (a.k.a. *oscillator algebra*) $\mathcal{A}$ is defined as a vector space to be $\mathcal{A}=\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}$ with the Lie bracket

$$\boxed{[(f,\alpha),(g,\beta)]=(0,\operatorname{Res}_{t=0} g\,df)} \quad \forall f,g\in\mathbb{C}[t,t^{-1}],\ \alpha,\beta\in\mathbb{C}$$

Down-to-earth: $\mathcal{A}$ has a basis $\{a_n\}_{n\in\mathbb{Z}}\cup\{K\}$ s.t.

$$\boxed{[K,a_n]=0\ ,\ [a_n,a_m]=n\delta_{n,-m}\cdot K} \quad \forall n,m\in\mathbb{Z}$$

(here: $K\longleftrightarrow(0,1)$, $a_n\longleftrightarrow(t^n,0)$)

To introduce Vir, we start first with another important Lie algebra:

Def 2: The *Witt algebra* W is the Lie algebra of polynomial vector fields

$$W=\{f(t)\partial_t \mid f\in\mathbb{C}[t,t^{-1}]\}$$

with the Lie bracket being the usual commutator of v. fields:

$$\boxed{[f(t)\partial_t,\ g(t)\partial_t]=(f(t)g'(t)-g(t)f'(t))\partial_t}$$

Down-to-earth: W has a basis $\{L_n\}_{n\in\mathbb{Z}}$ with the Lie bracket

$$\boxed{[L_n,L_m]=(n-m)L_{n+m}}$$

(here: $L_n\longleftrightarrow -t^{n+1}\partial_t$)

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Notably, the algebra $\overline{W}$ has a very important central extension (called the Virasoro algebra) which plays key role in Physics & V.O.A.

But, first, we shall discuss central extensions of Lie algebras.

Def 3: Let L be any Lie algebra. A 1-dim central extension of L is a Lie algebra $\hat{L}$ that fits into the following short exact sequence of Lie algebras:

ι - injective
π - surjective
$\operatorname{Ker}\pi = \operatorname{Im}\iota$

$$\boxed{0 \longrightarrow \underset{\text{trivial } [\,,\,]}{\mathbb{C}} \xrightarrow{\iota} \hat{L} \xrightarrow{\pi} L \longrightarrow 0}$$

We shall now discuss how to classify such extensions (up to equiv).

Step 1: Looking at the above sequence as just v. spaces, we can split $\hat{L} = L \oplus \mathbb{C}$ in a non-unique way. Then the Lie bracket on $\hat{L}$ is

$$[(a,\alpha),(b,\beta)] = ([a,b], \omega(a,b)) \quad \forall a,b \in L \quad \forall \alpha,\beta \in \mathbb{C}$$

as follows from ι & π being Lie algebra homomorphisms.

Here: ω defines a Lie bracket on $\hat{L} = L \oplus \mathbb{C}$ iff:

1) $\omega(a,b) = -\omega(b,a) \quad \forall a,b \in L$

2) ω satisfies the so-called "2-cocycle" condition

$$\omega([a,b],c) + \omega([b,c],a) + \omega([c,a],b) = 0$$

(indeed $[\,,\,]$ - skew $\Leftrightarrow \omega$ - skew
$[\,,\,]$ satisfies Jacobi $\Leftrightarrow$ ω - "2-cocycle")

Step 2: Note however that different splitting (as v. spaces) of $\hat{L}$ will give rise to different ω's, and we shall identify those i.e. we are interested in short exact sequence up to

$$\boxed{\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \hat{L}_{\omega_1} & \longrightarrow & L & \longrightarrow & 0 \\ & & \wr\downarrow \mathrm{Id} & & \wr\downarrow \varphi & & \wr\downarrow \mathrm{Id} & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \hat{L}_{\omega_2} & \longrightarrow & L & \longrightarrow & 0 \end{array}}$$

(Fig 1)

Note that φ must act as follows: $\varphi(a,\alpha) = (a, \alpha + \xi(a)) \quad \forall a \in L, \alpha \in \mathbb{C}$ where ξ is a linear map $\xi : L \to \mathbb{C}$

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Moreover, φ is a Lie algebra isomorphism iff

$$\varphi([(a,\alpha),(b,\beta)]_{\hat{L}_{\omega_1}}) = [\varphi(a,\alpha),\varphi(b,\beta)]_{\hat{L}_{\omega_2}} \quad \forall\, a,b\in L,\ \alpha,\beta\in\mathbb{C}$$

Here: $\varphi([(a,\alpha),(b,\beta)]_{\hat{L}_{\omega_1}}) = \varphi(([a,b],\omega_1(a,b))) = ([a,b],\omega_1(a,b)+\xi([a,b]))$

$[\varphi(a,\alpha),\varphi(b,\beta)]_{\hat{L}_{\omega_2}} = [(a,\alpha+\xi(a)),(b,\beta+\xi(b))]_{\hat{L}_{\omega_2}} = ([a,b],\omega_2([a,b]))$

Upshot: Two 2-cocycles give equivalent (in the above sense) central extensions iff $\exists\, \xi\in L^*$ s.t. $\omega_2(a,b)-\omega_1(a,b)=\xi([a,b])$

Thus, the 1-dim central extensions are parametrized by

$$\boxed{H^2(L) = Z^2(L)/B^2(L)} = 2^{nd} \text{ cohomology of } \mathfrak{g}$$

where $Z^2(L) = \{$2-cocycles (skew-symm) $\omega: \Lambda^2 L \to \mathbb{C}\}$

$\cup$

$B^2(L) = \{$2-coboundaries $\mid \omega(a,b)=\xi([a,b])$ for some $\xi\in L^*\}$

Remark: If we don't want to fix a basis of $\mathbb{C}$, then it's more natural to identify $\hat{L}_{\omega_1}$ & $\hat{L}_{\omega_2}$ whenever they fit into

$$\boxed{\begin{array}{ccccccccc} 0 & \to & \mathbb{C} & \to & \hat{L}_{\omega_1} & \to & L & \to & 0 \\ & & {\scriptstyle \lambda\,\mathrm{Id}}\downarrow\wr & & \downarrow\varphi & & {\scriptstyle ?}\downarrow \mathrm{Id} & & \\ 0 & \to & \mathbb{C} & \to & \hat{L}_{\omega_2} & \to & L & \to & 0 \end{array}}$$

(Fig 2)

Then, by above, nontrivial central extensions are parametrized by the projectivization $\mathbb{P}(H^2(L))$

The main result for today is:

Theorem 1: The space $H^2(W)$ is 1-dimensional, spanned by ω given by
$$\omega(L_n,L_m)=(n^3-n)\,\delta_{n,-m}$$

We will show that $H^2(W)$ is at most 1-dim, spanned by above ω.

Exercise (easy): Verify the above ω is a 2-cocycle but is not a 2-coboundary.

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Before presenting the proof, let us now introduce the Virasoro alg:

Def 4: The *Virasoro algebra* Vir is the 1-dim central extension of W defined by the 2-cocycle $\omega(L_n, L_m) = \frac{n^3-n}{12}\delta_{n,-m}$

Down-to-earth: Vir has a basis $\{L_n\}_{n\in\mathbb{Z}} \cup \{C\}$ s.t.

$$\boxed{[C, L_n] = 0, \quad [L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\cdot C}$$

Note: 1) The factor $1/12$ stands solely for historical reasons

2) By Thm 1, Vir is a unique nontrivial 1-dim central extension of W (up to equivalence in (Fig 2))

Proof of Theorem 1

▸ Pick any $\beta \in Z^2(W)$, i.e. β - 2-cocycle. We want to subtract some 2-coboundaries to obtain a multiple of the specified ω. To this end, we proceed in several steps:

Step 1

Let $\xi \in W^*$ be chosen to satisfy $\xi(L_n) = \frac{1}{n}\beta([L_n, L_0]) \quad \forall n\neq 0$, and set

$$\boxed{\tilde\beta(a,b) := \beta(a,b) - \xi([a,b])}$$

Key property: $\tilde\beta(L_n, L_0) = \begin{cases} \beta(L_n, L_0) - \xi(nL_n) = 0 & \forall n\neq 0 \\ \beta(L_0, L_0) - \xi(0) = 0 \end{cases} \Rightarrow \boxed{\tilde\beta(L_n, L_0) = 0 \quad \forall n}$

Step 2

Let's now write down the 2-cocycle condition for $\tilde\beta$ and $a=0, b=L_m, c=L_n$:

$$0 = \tilde\beta([L_0, L_m], L_n) + \tilde\beta([L_m, L_n], L_0) + \tilde\beta([L_n, L_0], L_m)$$
$$= -m\tilde\beta(L_m, L_n) + (m-n)\tilde\beta(L_{m+n}, L_0) + n\tilde\beta(L_n, L_m) \overset{\text{Step 1}}{=} (n+m)\tilde\beta(L_n, L_m)$$

So: $\boxed{\tilde\beta(L_n, L_m) = 0 \quad \text{if } n\neq -m.}$

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Step 3

Let $b_n = \tilde{\beta}(L_n, L_{-n})$, so that $b_n = b_{-n}$.

Then, the 2-cocycle condition for $\tilde{\beta}$ and $a = L_m$, $b = L_n$, $c = L_p$ is obvious unless $n+m+p=0$ (by Step 2), while plugging $a = L_m$, $b = L_n$, $c = L_{-m-n}$ $\forall m,n$

$$\tilde{\beta} \text{ - 2-cocycle} \iff \boxed{(n-m)\, b_{-m-n} - (m+2n)\, b_m + (2m+n)\, b_n = 0 \quad \forall n,m}$$

Step 4

Let $\xi' \in W^*$ be the map s.t. $\xi'(L_0) = 1$, $\xi'(L_n) = 0$ for $n \neq 0$, and set

$$\boxed{\tilde{\tilde{\beta}}(a,b) = \tilde{\beta}(a,b) - \frac{b_1}{2} \cdot \xi'([a,b])}$$

Key property:

$$\boxed{\tilde{\tilde{\beta}}(L_1, L_{-1}) = b_1 - \frac{b_1}{2} \cdot \xi'(2L_0) = 0}$$

Replacing $\tilde{\beta}$ by $\tilde{\tilde{\beta}}$ we can assume $b_1 = 0$ in Step 3. But then plugging $m=1$ over there gives:

$$(n-1)\, b_{-n-1} + (n+2)\, b_n = 0 \implies \boxed{b_{n+1} = \frac{n+2}{n-1}\, b_n}$$

So:

$$b_n = \frac{n+1}{n-2}\, b_{n-1} = \frac{n+1}{n-2} \cdot \frac{n}{n-3}\, b_{n-2} = \ldots = \frac{(n+1)\, n\, (n-1) \cdot \ldots \cdot 4}{(n-2) \cdot \ldots \cdot 4 \cdot 3 \cdot 2 \cdot 1}\, b_2 = \frac{n^3 - n}{6}\, b_2$$

Easy: For any b_2, the choice $b_n = \frac{n^3-n}{6}\, b_2$ $\forall n$ satisfies above $\forall m,n$.

∎

Next time: We will start by defining affine Kac-Moody algebras and prove a similar uniqueness for $\mathfrak{g}$-simple f.d.

Thus, it's recommended to familiarize yourself a bit with these alg-s (we will state all required results!)