Lecture #3

Last time: $\hat{\mathfrak{g}}$ as a unique nontrivial central extension of $\mathfrak{g}[t,t^{-1}]$ for $\mathfrak{g}$-simple f.d. $\mathcal{A}$-modules

Today: Finish $\mathcal{A}$-modules & start $\mathbb{Z}$-graded Lie algebras.

- Recall from last time

$$U(\mathcal{A})/(K-1) \xrightarrow{\sim} \mathrm{Diff}(x_1, x_2, \ldots) \otimes \mathbb{C}[x_0] \curvearrowright \underbrace{F_\mu}_{\text{Fock module}} = \mathbb{C}[x_1, x_2, \ldots]$$
$$x_0 \longmapsto \mu \cdot \mathrm{Id} \qquad \forall \mu \in \mathbb{C}.$$

Lemma 1: Modules $\{F_\mu \mid \mu \in \mathbb{C}\}$ are pairwise non-isomorphic irreducible $\mathcal{A}$-modules.

▸ As $a_0|_{F_\mu} = \mu \cdot \mathrm{Id}$, we clearly see $F_\mu \not\simeq F_\nu$ for $\mu \neq \nu$.

To prove that F_μ is irreducible, assume the contradiction, i.e. there is an $\mathcal{A}$-submodule $0 \subsetneq V \subsetneq F_\mu$. Take any $\overset{\neq 0}{v} = p(x_1, x_2, \ldots) \in V$, and let $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots$ be the monomial featuring in $p(x_1, x_2, \ldots)$ of the largest degree w.r.t. lexicographic order. Then applying $a_1^{n_1} a_2^{n_2} \ldots$ to v, we obtain a nonzero constant, hence $1 \in V$ (as V is an $\mathcal{A}$-submodule). But applying $\{a_{-j} \mid j > 0\}$ to 1, we get $V = F_\mu \Rightarrow$ ↯ ∎

In fact, as we shall now see, under subtle restrictions the Fock F_μ are the only irreducible $\mathcal{A}$-modules with $K \mapsto \mathrm{Id}$:

Proposition 1: a) Let V be an irreducible $\mathcal{A}$-module with $K \mapsto \mathrm{Id}_V$, $a_0 \mapsto \mu\, \mathrm{Id}_V$.

Assume: $\forall v \in V$, $\dim(\mathbb{C}[a_1, a_2, \ldots] v) < \infty$, and $\{a_j\}_{j>0}$ act by nilpotent operators on this fin. dim. subspace

Then: $V \simeq F_\mu$.

b) Let V be an $\mathcal{A}$-module, not necessarily irreducible, satisfying Assumptions of a) and s.t. $\forall v \in V\ \exists N$ s.t. $a_j(v) = 0\ \ \forall j > N$.

Then: $V \simeq F_\mu \otimes \underbrace{M}_{\text{vector space}}$, i.e. V is a direct sum of F_μ's

Before proceeding to the proof, let us make a simple observation:

Exercise: There is a bijection between $\mathcal{A}$-module homomorphisms (nonzero) $F_\mu \to V$ and $\{v \in V \mid a_j(v) = 0\ \forall j > 0,\ a_0(v) = \mu v,\ K(v) = v\}$

Lecture #3

Proof of Prop 1

a) Pick any nonzero $v \in V \setminus \{0\}$, and set $W = \mathbb{C}[a_1, a_2, \dots](v)$.

By assumption $\dim W < \infty$ and $\{a_j\}_{j>0}$ are pairwise commuting nilpotent op-s on W

Easy standard exercise: $\exists w \in W \setminus \{0\}$ s.t. $a_j(w) = 0$ $\forall j > 0$

(this is a simple case of Engel's Theorem; take $\mathrm{Ker}(a_1) \neq 0$ - it's stable under all other operators $\{a_j\}_{j \geq 2}$, next take $\mathrm{Ker}(a_1) \cap \mathrm{Ker}(a_2) \neq 0$ etc...)

Thus, we found $\overset{\neq 0}{w} \in V$ s.t. $K(w) = w$, $a_0(w) = \mu \cdot w$, $a_j(w) = 0$ $\forall j > 0$.

By Exercise on the page 1, this gives rise to a nonzero $\mathcal{A}$-module map

$$F_\mu \longrightarrow V \quad \text{with} \quad 1 \longmapsto w$$

But F_μ is irreducible (by Lemma 1) as well as V, hence, this map is isomorphism

b) The proof of part b) is a bit more technical, and involves a new ingredient

We start with

Claim 1: For any $v \in V$, the $\mathcal{A}$-submodule $\widetilde{W}$ generated by v has a finite length filtration with consecutive quotients $\simeq F_\mu$

(In other words, there is a family of $\mathcal{A}$-submodules

$$\boxed{0 = \widetilde{W}_0 \subsetneq \widetilde{W}_1 \subsetneq \widetilde{W}_2 \subsetneq \dots \subsetneq \widetilde{W}_{M-1} \subsetneq \widetilde{W}_M = \widetilde{W} \quad \text{s.t.} \quad \widetilde{W}_j / \widetilde{W}_{j-1} \simeq F_\mu \ \forall 1 \leq j \leq M}$$

Sketch of the proof

Let $I_v = \mathrm{Ann}(v) \subseteq \mathbb{C}[a_1, a_2, \dots]$ be the annihilator of v. As $\mathbb{C}[a_1, a_2, \dots]/I_v \xrightarrow{\sim} W$, see notations from a), we get I_v is a finite codimension subspace.

According to PBW thm, we have a surjective map

$$\boxed{\mathrm{Diff}(x_1, x_2, \dots) / (\mathrm{Diff}(x_1, x_2, \dots) \cdot I_v) \overset{\pi}{\twoheadrightarrow} \widetilde{W}}$$

Evoking the argument from a), since we have a pairwise commuting family of nilpotent operators $\{a_j\}_{j>0}$ acting on finite-dimensional W, and $a_j|_W = 0$ for $j \gg 1$ by our assumption, we can find a basis of W in which all $\{a_j\}$ act as strictly upper triangular matrices. This basis of $W \simeq \mathbb{C}[a_1, a_2, \dots]/I_v$ gives a complete flag of subspaces $I_v = J_M \subsetneq J_{M-1} \subsetneq \dots \subsetneq J_1 \subsetneq J_0 = \mathbb{C}[a_1, a_2, \dots]$ s.t.

$$\boxed{a_j(J_k) \subseteq J_{k+1} \quad \forall k < M \ \forall j > 0 \quad \text{and} \quad \mathrm{codim}(J_k) = k}$$

Lecture #3

(Continuation of the proof)

But then we set $D_j := \mathrm{Diff}(x_1, x_2, \dots) \cdot J_j \quad \forall\ 0 \le j \le M$ so that

$$\mathrm{Diff}(x_1, x_2, \dots) = D_0 \supseteq D_1 \supseteq D_2 \supseteq \dots \supseteq D_M = \mathrm{Diff}(x_1, x_2, \dots) \cdot I_v$$

which equips $\mathrm{Diff}(x_1, x_2, \dots) / \mathrm{Diff}(x_1, x_2, \dots) I_v$ with a finite filtration with consecutive quotients D_j / D_{j+1} being either zero or $\simeq F_\mu$.

Finally, taking π-images of this filtration, and erasing equal terms proves the Claim 1 $\square$

The rest of the proof of b) relies on the new ingredient

$\boxed{\text{Euler field } E = \sum_{j>0} a_{-j} a_j}$ (which lies in a completion of $\mathcal{U}(\mathcal{A})$)

As $\forall v \in V\ \exists N$ s.t. $a_j(v) = 0$ for $j > N$, we see that the action of E on V is well defined.

In particular, E acts via $\sum_{j>0} j x_j \partial_{x_j}$ on F_μ and is thus a diagonalizable operator with eigenvalues in $\mathbb{Z}_{\ge 0}$, as $\boxed{E(x_1^{n_1} x_2^{n_2} \dots) = (n_1 + 2n_2 + 3n_3 + \dots) x_1^{n_1} x_2^{n_2} \dots}$

Thus, $\forall v \in F_\mu$, $\dim(\mathbb{C}[E] v) < \infty$, i.e. E acts "locally finitely" on F_μ.

$\boxed{\text{Exercise}}$: Use above + Claim 1 to show that E acts locally finitely on V, and $V = \bigoplus_{\lambda \in \mathbb{Z}_{\ge 0}} V[\lambda]$, where $V[\lambda]$ = generalized λ-eigenspace of $E|_V$

Claim 2: $V[0] = \{v \in V \mid a_j(v) = 0\ \ \forall j > 0\} = \mathrm{Ker}(E)$

▷ If $a_j(v) = 0\ \forall j > 0$, then clearly $E(v) = 0 \Rightarrow v \in V[0]$.

It remains to show $V[0] \subseteq \{v \in V \mid a_j(v) = 0\ \forall j > 0\}$. Assume not, i.e. $\exists\, w \overset{\ne 0}{\in} V[0]$ s.t. $a_j(w) \ne 0$ for some $j > 0$.

But: $E a_j = \sum_{k>0} a_{-k} a_k a_j = \sum_{k \ne j} a_j a_{-k} a_k - j a_j = a_j (E - j)$, i.e. $\boxed{E \cdot a_j = a_j (E - j)}$ as operators on V.

Then $(E + j)^s a_j w = a_j E^s w = 0$ for $s \gg 1$ as $w \in V[0] \Rightarrow a_j(w) \in V[-j] = 0 \Rightarrow$ ↯ $\square$

Evoking Exercise from page 1, we thus get an $\mathcal{A}$-module morphism

$$\boxed{F_\mu \otimes \underbrace{V[0]}_{\text{multiplicity space}} \xrightarrow{\psi} V}$$

Lecture #3

We claim that this $\mathcal{A}$-module morphism ψ is an isomorphism

$\boxed{\text{Exercise}}$: ψ - injective

Let's explain the surjectivity of ψ. Assume it's not, i.e. $\overline{V} := V/\mathrm{Im}\,\psi$ is a nonzero $\mathcal{A}$-module. Applying the above argument to $\overline{V}$ instead of V, we know it contains a submodule $\simeq F_\mu$, hence, $\exists\, \overline{v} \in \overline{V} \setminus \{0\}$ s.t. $E(\overline{v}) = 0$.

On the other hand, by our definition, $E \curvearrowright \overline{V}$ with nonzero eigenvalues

Contradiction!

Remark: a) Claim 1 + Loc. Finiteness are standard tools to run usual arguments in ∞-dimensional setup

b) The Euler field played a key role due to $E|_{F_\mu}$ and $V[0]$ producing the desired multiplicity set. However, as a formal expression it's an infinite sum (which truncates to a finite sum on any $v \in V$)

Note that $\mathcal{A}$ is a graded Lie algebra (see discussion below) and F_μ is its $\mathbb{Z}$-graded module where we declare $\deg(x_j) = \deg(a_{-j}) = -j$. Note:

$$E(x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots) = (n_1 + 2n_2 + 3n_3 + \dots) x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots = -\deg(x_1^{n_1} x_2^{n_2} \dots)\, x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots$$

This allows to deduce

$$\boxed{\mathrm{Tr}_{F_\mu}(q^{-E}) = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots} = \prod_{k \geq 1} \frac{1}{1-q^k}}$$

to which we will return later on.

- The above is a specific case of general setup of $\mathbb{Z}$-graded Lie alg-s, which we shall now discuss.

Lecture #3

Def 1: A $\mathbb{Z}$-graded Lie algebra is a Lie algebra $\mathfrak{g}$ with a grading

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n \leftarrow \text{v. space decomposition}$$

such that $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ $\forall n,m$.

We shall be interested in a special class of such Lie algebras:

Def 2: A $\mathbb{Z}$-graded Lie algebra is called "non-degenerate" if:

1) $\dim(\mathfrak{g}_n) < \infty$ $\forall n$
2) $\mathfrak{g}_0$ is abelian, i.e. $[x,y]=0$ $\forall x,y \in \mathfrak{g}_0$
3) $\forall n \geq 1$ and "generic" $\lambda \in \mathfrak{g}_0^*$, the pairing $\mathfrak{g}_n \times \mathfrak{g}_{-n} \to \mathbb{C}$ is nondegenerate
$$(x, y) \mapsto \lambda([x,y])$$

Note: • 3) above implies $\dim(\mathfrak{g}_n) = \dim(\mathfrak{g}_{-n})$ $\forall n$.

• $\mathfrak{n}_+ := \bigoplus_{n>0} \mathfrak{g}_n$, $\mathfrak{n}_- := \bigoplus_{n<0} \mathfrak{g}_n$ – Lie subalgebras of $\mathfrak{g}$

• $\mathfrak{h} := \mathfrak{g}_0$ is a Lie subalg. of $\mathfrak{g}$, and is abelian by 2).

Def 3: The triangular decomposition of $\mathfrak{g}$ is a v.space decomposition
$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

These algebras include all examples we discussed last week:

Example: 1) $\mathcal{A}$ is $\mathbb{Z}$-graded via $\mathcal{A}_n = \begin{cases} \mathbb{C}a_n & \text{if } n\neq 0 \\ \mathbb{C}a_0 \oplus \mathbb{C}K & \text{if } n=0. \end{cases}$

2) W is $\mathbb{Z}$-graded via $W_n = \mathbb{C}L_n$

3) Vir is $\mathbb{Z}$-graded via $\mathrm{Vir}_n = \begin{cases} \mathbb{C}L_n & \text{if } n\neq 0 \\ \mathbb{C}L_0 + \mathbb{C}\cdot C & \text{if } n=0. \end{cases}$ ← central elt

4) $\mathfrak{g}$ - simple f.d. Lie algebra is $\mathbb{Z}$-graded via $\deg(e_i)=1$, $\deg(f_i)=-1$.
e.g. for $\mathfrak{g}=\mathfrak{sl}_n$, we have $\deg(E_{ij}) = j-i$, $\deg(\text{diagonal matrix})=0$.

5) Affine Kac-Moody $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$ is $\mathbb{Z}$-graded via above grading on $\mathfrak{g} = \mathfrak{g}\cdot t^0$ together with $\deg(e_\theta t^{-1}) = -1$, $\deg(f_\theta t) = 1$, $\deg(K)=0$, where θ is the longest root.

Exercise: Verify these are all non-degenerate $\mathbb{Z}$-graded Lie algebras while $\mathfrak{g}[t,t^{-1}]$ is degenerate (you may assume $\mathfrak{g}=\mathfrak{sl}_n$ in 4) & 5))

Lecture #3

Recall that a $\mathfrak{g}$-module ($\mathfrak{g}$ - Lie algebra) is the same as a $\mathcal{U}\mathfrak{g}$- module
universal enveloping

Def 4: Given a Lie algebra $\mathfrak{a}$, its Lie subalgebra $\mathfrak{b}$, and a $\mathfrak{b}$-module M the induced $\mathfrak{a}$-module $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M)$ is defined as

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M) = \mathcal{U}\mathfrak{a} \otimes_{\mathcal{U}\mathfrak{b}} M$$

With this at hand, we can now define Verma modules:

Def 5: Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra with $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\lambda \in \mathfrak{h}^*$.

a) The highest weight Verma module $M_\lambda = M_\lambda^+$ is

$$\boxed{M_\lambda^+ = \mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}(\mathbb{C}_\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda}$$

where $\mathbb{C}_\lambda$ is a 1dim module over $\mathfrak{h}\oplus\mathfrak{n}_+$ with $h.1 = \lambda(h)$, $x.1 = 0$ $\forall h \in \mathfrak{h}, x \in \mathfrak{n}_+$

b) The lowest weight Verma module M_λ^- is

$$\boxed{M_\lambda^- = \mathrm{Ind}_{\mathfrak{n}_-\oplus\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\mathfrak{n}_-\oplus\mathfrak{h})} \mathbb{C}_\lambda}$$

where $\mathbb{C}_\lambda$ is a 1dim $\mathfrak{n}_-\oplus\mathfrak{h}$- module with $h.1 = \lambda(h)$, $x.1 = 0$ $\forall h \in \mathfrak{h}, x \in \mathfrak{n}_-$

We shall study these modules next time!