Lecture #4

Last time: $\mathbb{Z}$-graded Lie algebras & Verma modules

Today: Further study of Verma modules

Let's start with the following classical result, left as an exercise:

$\boxed{\text{Exercise}}$ (Frobenius reciprocity). Let $\mathfrak{a}$ be a Lie algebra, $\mathfrak{b} \subseteq \mathfrak{a}$ - Lie subalg. Let $\mathfrak{b} \curvearrowright M$ and $\mathfrak{a} \curvearrowright N$. Then, we have
$$\mathrm{Hom}_{\mathfrak{a}}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M), N) = \mathrm{Hom}_{\mathfrak{b}}(M, N)$$
viewed as $\mathfrak{b}$-module here, also denoted $\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{a}}(N)$ = restriction of N to $\mathfrak{b}$

where $\mathrm{Hom}_{\mathfrak{g}}(-,-)$ means $\mathfrak{g}$-module maps

Def 1: For a $\mathfrak{g}$-module V, a vector $v \in V$ is called a singular vector of weight $\lambda \in \mathfrak{h}^*$ if $x(v)=0$, $h(v)=\lambda(h)\cdot v$ $\forall x \in \mathfrak{n}^+, h \in \mathfrak{h}$.

We denote the subspace of all such vectors by $\mathrm{sing}_\lambda(V)$

With this definition, one can define the highest weight Verma via the following universal property

Lemma 1: For any $\mathfrak{g}$-module V, there is a canonical isomorphism of v.sp:
$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, V) \xrightarrow{\sim} \mathrm{sing}_\lambda(V)$$

▸ $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, V) \overset{\text{Exercise}}{=\!=} \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda, V) \overset{\text{clear}}{=\!=} \mathrm{sing}_\lambda(V)$ ■

We also have the following simple property of Verma modules (here a module M of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}=\oplus\mathfrak{g}_n$ is called $\mathbb{Z}$-graded if $M=\bigoplus_{n\in\mathbb{Z}} M_n$ as v.spaces with $\mathfrak{g}_n(M_m) \subseteq M_{n+m}$ $\forall n,m$)

Lemma 2: (a) $M_\lambda^\pm$ are $\mathbb{Z}$-graded $\mathfrak{g}$-modules

(b) $M_\lambda^+ \simeq U(\mathfrak{n}_-)v_\lambda^+$ & $M_\lambda^- \simeq U(\mathfrak{n}_+)v_\lambda^-$ as vector spaces, where $v_\lambda^\pm \in M_\lambda^\pm$ is the image of $1\otimes 1^{\mathbb{C}_\lambda}$

▸ a) Obvious as $\mathfrak{n}_+$ acts trivially on $\mathbb{C}_\lambda$, while $\deg(\mathfrak{h}) = \deg(\mathfrak{g}_0)=0$

b) This follows easily from the PBW theorem. Let's give details.

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(continuation of proof of Lemma 2)

Pick a basis of $\mathfrak{g}$ as a union of bases of $\mathfrak{n}_-$, $\mathfrak{h}$, and $\mathfrak{n}_+$. Then, applying PBW to $U(\mathfrak{n}_-)$, $U(\mathfrak{h}\oplus\mathfrak{n}_+)$, and $U(\mathfrak{g})$, we see that

multiplication $U(\mathfrak{n}_-)\otimes U(\mathfrak{h}\oplus\mathfrak{n}_+)\longrightarrow U(\mathfrak{g})$ is a v.space iso.

Hence: $M_\lambda^+ = U(\mathfrak{g})\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)}\mathbb{C}_\lambda \cong (U(\mathfrak{n}_-)\otimes U(\mathfrak{h}\oplus\mathfrak{n}_+))\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)}\mathbb{C}_\lambda \simeq U(\mathfrak{n}_-)\otimes_{\mathbb{C}}\mathbb{C}_\lambda$

Likewise: $M_\lambda^- \simeq U(\mathfrak{n}_+)\otimes_{\mathbb{C}}\mathbb{C}_\lambda$ as v. spaces.

Actually: $M_\lambda^\pm \simeq U(\mathfrak{n}_\mp)\otimes_{\mathbb{C}}\mathbb{C}_\lambda$ as graded $\mathfrak{n}_\mp$-modules ■

As an immediate corollary, we get:

Corollary 1: a) If $\dim(\mathfrak{g}_{-n})<\infty$ $\forall n>0$, then $\sum_{n\geq 0}\dim(M_\lambda^+[-n])\cdot q^n = \prod_{k\geq 1}\frac{1}{(1-q^k)^{\dim \mathfrak{g}_{-k}}}$

b) If $\dim(\mathfrak{g}_n)<\infty$ $\forall n>0$, then $\sum_{n\geq 0}\dim(M_\lambda^-[n])\cdot q^n = \prod_{k\geq 1}\frac{1}{(1-q^k)^{\dim \mathfrak{g}_k}}$

► This follows from PBW as $U(\mathfrak{n}_\mp)\simeq \underbrace{S(\mathfrak{n}_\mp)}_{\text{Symmetric algebra of } \mathfrak{n}_\mp}$ as graded vector spaces ■

Example: Consider $\mathfrak{g}=\mathcal{A}$ - Heisenberg algebra, $\mathbb{Z}$-graded as in Lecture 3.
Then $\mathfrak{h}=\mathfrak{g}_0=\mathbb{C}a_0\oplus\mathbb{C}K$. Pick $\lambda\in\mathfrak{h}^*$ s.t. $\lambda(K)=1$, $\lambda(a_0)=\mu$.
Then, $M_\lambda^+ \xrightarrow{\sim} F_\mu$ = Fock $\mathcal{A}$-module from Lecture 2 (check it!)

The main result from today establishes an interplay b/w M_λ^+ & M_μ^-.
We start with two definitions:

Def 2: Given $\mathbb{Z}$-graded vector spaces $M=\bigoplus_m M_m$, $N=\bigoplus_n N_n$, a bilinear form $(\,,\,): M\times N\longrightarrow\mathbb{C}$ is *of degree zero* if $(M_m, N_n)=0$ for $m+n\neq 0$

Def 3: Given $\mathfrak{g}$-modules M, N, a bilinear form $(\,,\,): M\times N\longrightarrow\mathbb{C}$ is called *$\mathfrak{g}$-invariant* if $(xm,n)+(m,xn)=0$ $\forall m\in M, n\in N, x\in\mathfrak{g}$

The latter can be interpreted as saying $(\,,\,): M\otimes N\longrightarrow\mathbb{C}$ is a $\mathfrak{g}$-module morphism, where $\mathbb{C}$ is the trivial $\mathfrak{g}$-module

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The following is our first main result for today:

Proposition 1: Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra and $\lambda \in \mathfrak{h}^*$. Then, there exists a unique (up to scaling) $\mathfrak{g}$-invariant pairing $M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$. Moreover, it is of degree zero

The proof relies on the following exercise (see Homework 1):

Exercise: (a) Let $\mathfrak{a}$ - Lie algebra, $\mathfrak{b} \subseteq \mathfrak{a}$ - Lie subalgebra, $\mathfrak{b} \curvearrowright M$, $\mathfrak{a} \curvearrowright N$.

Then: $\quad \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{b}}(M) \otimes N \simeq \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{b}}(M \otimes \underbrace{\mathrm{Res}^{\mathfrak{a}}_{\mathfrak{b}}(N)}_{\text{same } N \text{ viewed as } \mathfrak{b}\text{-module}})$ as $\mathfrak{a}$-modules

(b) Let $\mathfrak{c}$ be a Lie algebra, $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{c}$ - Lie subalgebras s.t. $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$

Then for any $\mathfrak{b} \curvearrowright M$, we have

$$\mathrm{Res}^{\mathfrak{c}}_{\mathfrak{a}}(\mathrm{Ind}^{\mathfrak{c}}_{\mathfrak{b}}(M)) \simeq \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{a}\cap\mathfrak{b}}(\mathrm{Res}^{\mathfrak{b}}_{\mathfrak{a}\cap\mathfrak{b}}(M)) \text{ as } \mathfrak{a}\text{-modules}$$

Proof of Proposition 1

$$\mathrm{Hom}_{\mathfrak{g}}(M^+_\lambda \otimes M^-_{-\lambda}, \mathbb{C}) \overset{\text{Ex a)}}{=\!=} \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes M^-_{-\lambda}), \mathbb{C}) \overset{\text{Frob. reciprocity}}{=\!=}$$

$$= \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes M^-_{-\lambda}, \mathbb{C}) = \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_-}(\mathbb{C}_{-\lambda})), \mathbb{C})$$

$$\overset{\text{Ex b)}}{=\!=} \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes \mathrm{Ind}^{\mathfrak{h}\oplus\mathfrak{n}_+}_{\mathfrak{h}}(\mathbb{C}_{-\lambda}), \mathbb{C}) \overset{\text{Ex a)}}{=\!=} \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathrm{Ind}^{\mathfrak{h}\oplus\mathfrak{n}_+}_{\mathfrak{h}}(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}), \mathbb{C})$$

$$\overset{\text{Frob. reciprocity}}{=\!=} \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}, \mathbb{C}) \simeq \mathbb{C}$$

Moreover, the resulting map $\mathrm{Hom}_{\mathfrak{g}}(M^+_\lambda \otimes M^-_{-\lambda}, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}$ sends $(\,,\,) \mapsto (v^+_\lambda, v^-_{-\lambda})$

Let's show that such $(\,,\,)$ is necessarily of degree zero. To this end we want to verify $(xv^+_\lambda, yv^-_{-\lambda}) = 0$ for any $x \in U(\mathfrak{n}_-)$, $y \in U(\mathfrak{n}_+)$ with $\deg(x) = -n$, $\deg(y) = m$ and $n + m \neq 0$. Let $S: U(\mathfrak{g}) \to U(\mathfrak{g})$ be the algebra antiautomorphism s.t. $S(z) = -z$ $\forall z \in \mathfrak{g}$ (called antipode map)

$\mathfrak{g}$-invariance of $(\,,\,)$ implies $\quad (xv^+_\lambda, yv^-_{-\lambda}) = (S(y)xv^+_\lambda, v^-_{-\lambda}) = (v^+_\lambda, S(x)yv^-_{-\lambda})$

If $m - n > 0 \Rightarrow S(y)xv^+_\lambda = 0$; if $m - n < 0 \Rightarrow S(x)yv^-_{-\lambda} = 0$.

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<u>Remark</u>: The above proof also shows that if $\mu \neq -\lambda \in \mathfrak{h}^*$, then there are <u>no</u> $\mathfrak{g}$-invariant pairings $M^+_\lambda \times M^-_\mu \to \mathbb{C}$.

<u>Def 4</u>: Let $(\,,\,)_\lambda$ denote the $\mathfrak{g}$-inv. pairing of Prop 1 s.t. $(v^+_\lambda, v^-_{-\lambda}) = 1$

Let J^+_λ (resp. $J^-_{-\lambda}$) be the left (resp. right) <u>kernel</u> of $(\,,\,)_\lambda$, i.e.

$$J^+_\lambda = \{v \in M^+_\lambda \mid (v, M^-_{-\lambda})_\lambda = 0\}, \quad J^-_{-\lambda} = \{v \in M^-_{-\lambda} \mid (M^+_\lambda, v) = 0\}$$

Since $(\,,\,)_\lambda$ has degree zero, we see that $J^\pm_{\pm\lambda}$ are $\mathbb{Z}$-graded.
Since $(\,,\,)_\lambda$ is $\mathfrak{g}$-invariant, $J^\pm_{\pm\lambda}$ are also $\mathfrak{g}$-submodules
(indeed if $v \in J^+_\lambda$, $x \in \mathfrak{g}$, then $(xv, w) = -(v, xw) = 0 \ \forall w \Rightarrow xv \in J^+_\lambda$)
Thus, we obtain $\mathbb{Z}$-graded $\mathfrak{g}$-modules

$$\boxed{L^\pm_\lambda := M^\pm_\lambda / J^\pm_\lambda}$$

Moreover, by construction, the pairing $(\,,\,)_\lambda$ descends to a

$$\boxed{\text{non-degenerate } \mathfrak{g}\text{-invariant degree zero pairing } (\,,\,)_\lambda : L^+_\lambda \times L^-_{-\lambda} \to \mathbb{C}}$$

<u>Theorem 1</u>: (a) $L^\pm_\lambda$ are irreducible $\mathfrak{g}$-modules

(b) $J^\pm_\lambda$ - maximal proper graded submodule of $M^\pm_\lambda$

(c) If there is $L \in \mathfrak{h}$ s.t. $[L, x] = n \cdot x \ \forall n \in \mathbb{Z} \ \forall x \in \mathfrak{g}_n$, then $J^\pm_\lambda$ - max. proper submodule of $M^\pm_\lambda$

<u>Reminder</u>: A subspace W of a $\mathbb{Z}$-graded v. space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is called <u>graded</u> if $W = \bigoplus_n (W \cap V_n)$. Same applies to submodules.

<u>Exercise</u>: For each $\mathfrak{g}$ from Lecture 2 (i.e. $\mathcal{A}$, W, Vir, $\mathfrak{g}$-simple, $\hat{\mathfrak{g}}$) decide if L as in part (c) exists

<u>Exercise</u>: Provide a counterexample to (b) if "graded" is lifted.

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Proof of Theorem 1

(a) Assume contradiction, i.e. $\exists\, \mathfrak{g}$-submodule $0 \neq V \subsetneq L^+_\lambda$. Take any $v \in V \setminus 0$.

Write $v = \sum_{i=0}^{N} v_{-i}$, v_{-i} has degree $-i$, $v_{-N} \neq 0$. For any $x \in \mathfrak{g}_j$ $(j>0)$:

$xv = \sum \underbrace{x v_{-i}}_{\text{degree } j-i}$ is an elt of V, with minimal degree component $> -N$.

Thus, applying n^+ several times we can assume that v is chosen so that N-minimal possible. Then $n_+(v_{-N}) = 0$. Also $N \neq 0$ (as then $V = L^+_\lambda \Rightarrow$ ↯).

But $\mathfrak{g}$-invariance of $(\,,\,)_\lambda$ then implies $(v_{-N}, -) = 0 \Rightarrow v_{-N} \in J^+_\lambda \Rightarrow \underset{\text{in } L^+_\lambda}{v_{-N} = 0}$

Contradiction! Proof for L^-_λ is analogous.

(b) Let $V \subseteq M^+_\lambda$ be a graded $\mathfrak{g}$-submodule. If $V \not\subseteq J^+_\lambda$, then its image $\bar{V}$ in $M^+_\lambda / J^+_\lambda = L^+_\lambda$ is a nonzero submodule. Moreover, $\bar{V} \neq L^+_\lambda$ as $\bar{V}$ is $\mathbb{Z}$-graded and has zero degree $=0$ component. Contradiction with a).

Proof for J^-_λ is analogous.

(c) It suffices to show that any $\mathfrak{g}$-submodule V of M^+_λ is necessarily $\mathbb{Z}$-graded given the condition about $L \in \mathfrak{h}$. Indeed, take any $v \in V \setminus \{0\}$ written as $v = \sum_{i=0}^{N} v_{-i}$, $\deg(v_{-i}) = -i$. Need to show $v_{-i} \in V$

But applying $\mathrm{ad}(-L)^k$ to v, we get $\underbrace{\mathrm{ad}(-L)^k v}_{\in V} = \sum_{i=0}^{N} i^k \cdot v_{-i}$

Using Vandermonde determinant f-la, we now get $v_{-i} \in V$ as claimed ∎

Our last result for today (which we shall outline next time) is:

Theorem 2: If $\mathfrak{g}$ is a non-degenerate $\mathbb{Z}$-graded Lie algebra, then for any $n \geq 1$, the form $(\cdot,\cdot)_\lambda : M^+_\lambda[-n] \times M^-_{-\lambda}[n] \to \mathbb{C}$ is non-degenerate for "generic" $\lambda \in \mathfrak{h}^*$

Thm 2 + Thm 1(c) imply:

Corollary 2: For non-deg. $\mathbb{Z}$-graded $\mathfrak{g}$, $M^\pm_\lambda$ are irreducible for generic λ