Lecture #5

Last time: • $\mathfrak{g}$-invariant form $(\, , \,)_\lambda : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ s.t. $(v^+_\lambda, v^-_{-\lambda})_\lambda = 1$

• $L^\pm_\lambda = M^\pm_\lambda / J^\pm_\lambda$ with $J^+_\lambda \subseteq M^+_\lambda$, $J^-_{-\lambda} \subseteq M^-_{-\lambda}$ being the kernels of $(\, , \,)_\lambda$.
↑ irred.

• Stated Thm 2 which we shall recall today

Today: Continuation of this + Category $\mathcal{O}^\pm$.

• Let us start from the practical side of $\mathfrak{g}$-inv. form. Note that $(\, , \,): M \times N \to \mathbb{C}$ is $\mathfrak{g}$-invariant if $(xm, n) = (m, (-x)\cdot n)$ $\forall m \in M, n \in N, x \in \mathfrak{g}$.
Now if we take several $x_1, \ldots, x_k \in \mathfrak{g}$, then $(x_1 \ldots x_k m, n) = (m, (-x_k) \cdot \ldots \cdot (-x_1) n)$.
This allows to write $\mathfrak{g}$-invariance for $x \in \mathcal{U}(\mathfrak{g})$.

Def 1: Let $S: \mathcal{U}\mathfrak{g} \circlearrowleft$ be the antiautomorphism s.t. $S(x_1 \ldots x_k) = (-x_k) \cdot \ldots \cdot (-x_2)(-x_1)$ for any $x_1, \ldots, x_k \in \mathfrak{g}$ (called the antipode map).

Then: $\boxed{(\, , \,) \text{ is } \mathfrak{g}\text{-invariant iff } (xm, n) = (m, S(x) n) \ \forall x \in \mathcal{U}\mathfrak{g}.}$

• Present the proof of $(\, , \,)_\lambda$ being of degree zero – see p. 3 of Lecture #4
(but we forgot it in the class)

• Let's now state the last theorem from the previous class:

> **Theorem 1**: If $\mathfrak{g}$ is a non-degenerate $\mathbb{Z}$-graded Lie algebra, then $\forall n \geq 1$ the restriction $(\cdot , \cdot)_\lambda : M^+_\lambda[-n] \times M^-_{-\lambda}[n] \to \mathbb{C}$ is non-degenerate for "generic" $\lambda \in \mathfrak{h}^*$.

Before outlining the proof, let's see how one can view this in the simplest cases:

• $\mathfrak{g} = \mathfrak{sl}_2$, n – any

• $\mathfrak{g} = Vir$, $n = 1$ or 2.

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Example 1: Let $\mathfrak{g}=\mathfrak{sl}_2$ with the standard basis $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ and the grading $\deg(e)=1$, $\deg(h)=0$, $\deg(f)=-1$. Identify $\mathfrak{h}^*\simeq\mathbb{C}$. As $\mathfrak{n}_-=\mathbb{C}f$, we get M^+_λ has a basis $\{f^n v^+_\lambda\}_{n\geq 0}$. Likewise $\{e^n v^-_{-\lambda}\}_{n\geq 0}$ is a basis of $M^-_{-\lambda}$. Thus: $M^+_\lambda[-n]$ is 1-dim spanned by $f^n v^+_\lambda$,
$M^-_{-\lambda}[n]$ is 1-dim spanned by $e^n v^-_{-\lambda}$.

Let's compute the corresponding restriction of $(\cdot,\cdot)_\lambda$ to these degrees:

$$(f^n v^+_\lambda, e^n v^-_{-\lambda})_\lambda = ((-1)^n e^n f^n v^+_\lambda, v^-_{-\lambda})_\lambda = (-1)^n\cdot n!\cdot\lambda(\lambda-1)\cdot\ldots\cdot(\lambda-n+1)$$

(this is based on the equality $ef^n v^+_\lambda = n(\lambda-n+1)f^{n-1}v^+_\lambda$ (+ induction on n) which is proved by induction "moving" e to the right of f and using also $hf^{n-1}v^+_\lambda=(\lambda-2(n-1))f^{n-1}v^+_\lambda$:

$$ef^n v^+_\lambda = fef^{n-1}v^+_\lambda + hf^{n-1}v^+_\lambda = f\cdot(n-1)(\lambda-n+2)f^{n-2}v^+_\lambda + (\lambda-2(n-1))f^{n-1}v^+_\lambda = n(\lambda-n+1)f^{n-1}v^+_\lambda\text{)}$$

<u>Upshot</u>: The restriction of $(\ ,\ )_\lambda$ to degree $\mp n$ pieces is given by a polyn-l:

$$(-1)^n\, n!\cdot\lambda(\lambda-1)\ldots(\lambda-n+1)$$

whose <u>leading term</u> is $(-1)^n\cdot n!\cdot\lambda^n$

As we shall see later today this implies the classical result:

<u>Corollary 1</u>: For $\mathfrak{g}=\mathfrak{sl}_2$, the Verma module M^+_λ is irreducible iff $\lambda\notin\mathbb{Z}_{\geq 0}$.
For $\lambda\in\mathbb{Z}_{\geq 0}$, $J^+_\lambda\subseteq M^+_\lambda$ defined last time is given by $J^+_\lambda=\mathrm{span}\{f^n v^+_\lambda \mid n>\lambda\}$ and so by Thm 1 of Lecture 4, $L^+_\lambda = M^+_\lambda/J^+_\lambda$ is $(\lambda+1)$-dim irreducible $\mathfrak{sl}_2$-module

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Example 2: $\mathfrak{g}=Vir$ with $\mathfrak{g}_n=\mathbb{C}L_n$ for $n\neq 0$, $\mathfrak{g}_0=\mathfrak{h}=\mathbb{C}L_0\oplus\mathbb{C}\cdot C$. Thus $\lambda\in\mathfrak{h}^*$ is determined by two numbers:

$h:=\lambda(L_0)$ – "conformal weight", $c:=\lambda(C)$ – "central charge"

$n=1$: Note that $M^+_\lambda[-1]$ is 1-dim spanned by $L_{-1}v^+_\lambda$, $M^-_{-\lambda}[1]$ – spanned by $L_1v^-_{-\lambda}$.

$$(L_{-1}v^+_\lambda, L_1v^-_{-\lambda})_{\lambda=(h,c)}=(-L_1L_{-1}v^+_\lambda, v^-_{-\lambda})_{h,c}=-(L_{-1}\underbrace{L_1v^+_\lambda}_{=0}+2\underbrace{L_0v^+_\lambda}_{h\cdot v^+_\lambda}, v^-_{-\lambda})_{h,c}=-2h.$$

So: restriction of $(\,,\,)_{h,c}$ to degrees ∓ 1 is non-degenerate iff $h\neq 0$.

$n=2$: In this case clearly $M^+_\lambda[-2]$ is 2-dim spanned by $L_{-2}v^+_\lambda, L_{-1}^2v^+_\lambda$

$M^-_\lambda[2]$ is 2-dim spanned by $L_2v^-_{-\lambda}, L_1^2v^-_{-\lambda}$.

Hence, restriction of $(\,,\,)_{h,c}$ to degrees ∓ 2 is given in that basis by 2x2 matrix

$$\begin{pmatrix}(L_{-1}^2v^+_\lambda, L_1^2v^-_{-\lambda}) & (L_{-1}^2v^+_\lambda, L_2v^-_{-\lambda})\\ (L_{-2}v^+_\lambda, L_1^2v^-_{-\lambda}) & (L_{-2}v^+_\lambda, L_2v^-_{-\lambda})\end{pmatrix}=\begin{pmatrix}8h^2+4h & -6h\\ 6h & -4h-\frac{c}{2}\end{pmatrix}$$

Compute: • $(L_{-2}v^+_\lambda, L_2v^-_{-\lambda})=-(L_2L_{-2}v^+_\lambda, v^-_{-\lambda})=-(L_{-2}\underbrace{L_2v^+_\lambda}_{=0}+4\underbrace{L_0v^+_\lambda}_{h\cdot v^+_\lambda}+\frac{2^3-2}{12}\underbrace{Cv^+_\lambda}_{c\cdot v^+_\lambda}, v^-_{-\lambda})$
$=-4h-c/2$

• $(L_{-1}^2v^+_\lambda, L_2v^-_{-\lambda})=-(L_2L_{-1}^2v^+_\lambda, v^-_{-\lambda})=-(L_{-1}\underbrace{L_2L_{-1}v^+_\lambda}_{L_{-1}L_2v^+_\lambda+3L_1v^+_\lambda=0}+3L_1L_{-1}v^+_\lambda, v^-_{-\lambda})=$
$=-3(L_1L_{-1}v^+_\lambda, v^-_{-\lambda})\overset{\text{above}}{=}-6h$

• $(L_{-1}^2v^+_\lambda, L_1^2v^-_{-\lambda})=(L_1^2L_{-1}^2v^+_\lambda, v^-_{-\lambda})=((L_1L_{-1}L_1L_{-1}+2L_1L_0L_{-1})v^+_\lambda, v^-_{-\lambda})$
$=(L_1L_{-1}^2\underbrace{L_1v^+_\lambda}_{=0}+2L_1L_{-1}\cdot\underbrace{L_0v^+_\lambda}_{h\cdot v^+_\lambda}+2L_1L_{-1}\underbrace{L_0v^+_\lambda}_{h\cdot v^+_\lambda}+2L_1L_{-1}v^+_\lambda, v^-_{-\lambda})$
$=(4h+2)\cdot(L_1L_{-1}v^+_\lambda, v^-_{-\lambda})\overset{\text{above}}{=}(4h+2)\cdot 2h=8h^2+4h$

Thus: $(\,,\,)_{h,c}$ restriction to ∓ 2 degree components is non-degenerate iff $\underbrace{\det\begin{pmatrix}8h^2+4h & -6h\\ 6h & -4h-\frac{c}{2}\end{pmatrix}}_{=-4h((2h+1)(4h+\frac{c}{2})-9h)}\neq 0$, i.e. if $(h,c)\notin$ (line $h=0$) $\cup$ {zeros of quadric $(2h+1)(4h+\frac{c}{2})-9h$}

$=-4h((2h+1)(4h+\frac{c}{2})-9h)$ ← leading term is $8h^2\cdot(-4h-\frac{c}{2})$.

Remark: One of the key results later in the course will be determinant f-la $\forall n\geq 1$.

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Sketch of the proof of Theorem 1

Recall that $M_\lambda^+ \simeq \mathcal{U}(n_-)$ as graded vector spaces, so that we have v.space isomorphism $\mathcal{U}(n_-)[-n] \xrightarrow{\sim} M_\lambda^+[-n]$ given by $a \mapsto a v_\lambda^+$. Likewise, $\mathcal{U}(n_+)[n] \xrightarrow{\sim} M_{-\lambda}^-[n]$, $b \overset{\psi}{\mapsto} b(v_{-\lambda}^-)$.
Thus, the restriction of $(\, , \,)_\lambda$ to degrees $\mp n$ components can be viewed as:

$$\boxed{(\, , \,)_{\lambda,n} : \mathcal{U}(n_-)[-n] \times \mathcal{U}(n_+)[n] \longrightarrow \mathbb{C} \quad , \quad (a,b) \mapsto (a v_\lambda^+, b v_{-\lambda}^-)_\lambda}$$

We can compute it as in above examples via $(a v_\lambda^+, b v_{-\lambda}^-)_\lambda = (S(b) a \, v_\lambda^+, v_{-\lambda}^-)_\lambda$ and then "moving" positive el-s (from $\mathcal{U}(n_+)$) to the right of negative el-s $\in \mathcal{U}(n_-)$)

(As $n_+(v_\lambda^+)=0$, what we basically do is taking elt $S(b)a \in \mathcal{U}(\mathfrak{g})$, writing it in the triangular decomposition $\mathcal{U}(n_-) \cdot \mathcal{U}(\mathfrak{h}) \cdot \mathcal{U}(n_+) \simeq \mathcal{U}(\mathfrak{g})$, and then projecting onto $\mathcal{U}(\mathfrak{h})$ (as $\deg(S(b)a)=0$, $S(b)a \in \mathcal{U}(\mathfrak{h}) \oplus \bigoplus_{k>0} \mathcal{U}(n_-)[-k] \cdot \mathcal{U}\mathfrak{h} \cdot \mathcal{U}(n_+)[k]$))

Picking any basis of $\mathfrak{h}$, say $h_1, \ldots, h_r$, at the end we get polynomial in $\{h_i\}$ acting on v_λ^+, but $h_i \cdot v_\lambda^+ = \lambda(h_i) \cdot v_\lambda^+$. This shows that each pairing $(a v_\lambda^+, b v_{-\lambda}^-)_\lambda$ is a polynomial in $\{\lambda(h_i)\}_{1 \le i \le r}$.

As $\mathfrak{g}$ - non-degenerate $\Rightarrow \dim(\mathfrak{g}_k) = \dim(\mathfrak{g}_{-k})$ $\forall k$ $\Rightarrow$ pick bases of each $\mathfrak{g}_k$ which by PBW gives bases of $\mathcal{U}(n_\pm)[\pm n]$. This produces explicit square matrix by writing $(\, , \,)_{\lambda,n}$ in that basis, with entries polynomials in $\{\lambda(h_i)\}$.
In particular, $(\, , \,)_{\lambda,n}$ is non degenerate $\iff$ det(this matrix) $\neq 0$
(note that the latter condition is invariant of the choice of bases!)

<u>So</u>: It remains to show $\det(\cdot,\cdot)_{\lambda,n} \neq 0$ for "generic" $\lambda \in \mathfrak{h}^*$.

<u>Strategy</u>: Degenerate Lie algebra $\mathfrak{g}$ to so-called "generalized Heisenberg algebra" so that the respective matrix for it depicts the leading term of above $\det(\cdot,\cdot)_{\lambda,n}$ and verify it's nonzero given $\mathfrak{g}$ - non-degenerate

this will be a *-Problem on Hwk 2.

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Proposition 1: M_λ^+ is irreducible iff it does not have singular vectors in <0 degrees (i.e. no singular vectors in $\oplus_{n>0} M_\lambda^+[-n]$)

($\Rightarrow$) If there is a singular vector $v \in M_\lambda^+$ as above, then by Lemma 1 of Lect 4, we get a nonzero $\mathfrak{g}$-module morphism $\phi: M_\mu^+ \to M_\lambda^+$, $v_\mu^+ \mapsto v$, with $\mu(h)\cdot v = h\cdot v$ $\forall h \in \mathfrak{h}$.

But then $\mathrm{Im}(\phi)$ is a nonzero submodule of M_λ^+ which is $\neq M_\lambda^+$ as $\mathrm{Im}(\phi) \cap M_\lambda^+[0] = 0$. This contradicts irreducibility of M_λ^+.

Exercise: Any nonzero $\mathfrak{g}$-module morphism $M_\mu^+ \to M_\lambda^+$ is injective.

($\Leftarrow$) Assuming contradiction (i.e. M_λ^+ is not irreducible), we see that $\exists\, v \in M_\lambda^+$ s.t. $\mathcal{U}(\mathfrak{g})v \subsetneq M_\lambda^+$

Exercise: Show that one can choose this v to be homogeneous ($\in M_\lambda^+[-n]$, $n>0$).

Then $\mathcal{U}(\mathfrak{g})v \subsetneq M_\lambda^+$ is a proper graded submodule of M_λ^+, hence, by Theorem 1(b) from last time: $\mathcal{U}(\mathfrak{g})v \subseteq J_\lambda^+ \Rightarrow J_\lambda^+$ is nonzero!

But J_λ^+ is graded: $J_\lambda^+ = \bigoplus_{d<0} J_\lambda^+[d]$, with $J_\lambda^+[d] = J_\lambda^+ \cap M_\lambda^+[d]$.

Picking largest $d<0$ s.t. $J_\lambda^+[d] \neq 0$, we see $\mathfrak{n}_+(J_\lambda^+[d]) = 0$, while $\mathfrak{h} \curvearrowright \underbrace{J_\lambda^+[d]}_{\text{fin. dim}}$ by pairwise commuting elements $\Rightarrow \exists\, \hat{v} \in J_\lambda^+[d]\setminus\{0\}$ common $\mathfrak{h}$-eigenvector.

Thus: $\hat{v}$ is a singular vector in M_λ^+ in <0 degrees $\Rightarrow$ Contradiction! $\blacksquare$

Remark: The Verma module M_λ^+ is not irreducible iff $\exists\, k>0$ and a singular vector $v \in M_\lambda^+[-k]$. Moreover, as the proof shows, the smallest of such k is precisely the smallest $k>0$ s.t. the determinant $\det(\cdot,\cdot)_{\lambda,k} = 0$, see p. 4 and Examples 1-2.

Note: By above Rmk, to check if M_λ^+ is irreducible it's key to have f-las for all these $\det(\cdot,\cdot)_{\lambda,k}$. We will discuss those for $\mathfrak{g} = Vir$, affine Kac-Moody

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Let us conclude today's class with the following important definition:

Def 2: The objects of category $\mathcal{O}^+$ are $\mathbb{C}$-graded $\mathfrak{g}$-modules

$$V = \bigoplus_{d\in\mathbb{C}} V[d] \qquad \mathfrak{g}_n(V[d]) \subseteq V[d+n] \quad \forall n\in\mathbb{Z}, d\in\mathbb{C}$$

such that:

a) $V[d]$ is finite dimensional $\forall d\in\mathbb{C}$

b) the set $\{d\in\mathbb{C} \mid V[d]\neq 0\}$ is contained in a finite union of arithmetic progressions with step -1, "bounded from right", i.e. $\exists N, d_1,\ldots,d_N\in\mathbb{C}$, s.t.

$$\{d\in\mathbb{C} \mid V[d]\neq 0\} \subseteq \bigcup_{i=1}^{N} \{d_i, d_i-1, d_i-2, \ldots\}$$

and the morphisms in the category $\mathcal{O}^+$ are graded $\mathfrak{g}$-module morphisms.

Replacing $\{d_i, d_i-1, d_i-2,\ldots\}$ by $\{d_i, d_i+1, d_i+2, \ldots\}$ one gets category $\mathcal{O}^-$.

Assumption: We shall assume that $\dim(\mathfrak{g}_n)<\infty\ \forall n$, $\mathfrak{g}_0 = \mathfrak{h}$ – abelian.

Example: $M^{\pm}_{\lambda}\in\mathcal{O}^{\pm}\ \forall \lambda\in\mathfrak{h}$, where we can place $v^{\pm}_{\lambda}$ in any degree $e\in\mathbb{C}$. (in particular, all $\{d \mid M^{\pm}_{\lambda}[d]\neq 0\}$ are then $\{e, e\mp 1, e\mp 2, \ldots\}$)

Thus any graded quotient, such as $L^{\pm}_{\lambda}$, is also in $\mathcal{O}^{\pm}$.

Remark: a) The category $\mathcal{O}$ is introduced precisely to include all Vermas, and to be closed under Ker, Coker

b) For those familiar with category $\mathcal{O}$ for simple f.dim. $\mathfrak{g}$, this definition is not precisely the same, but it plays the same role. In particular, insted of having cones in $\mathfrak{h}^*$ we just look at 1dim cones in $\mathbb{C}$, and this passage is given by taking the principal grading (i.e. pairing with $\rho^\vee$)

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<u>Proposition 2</u>: $L^\pm_\lambda$ are the only irreducible modules in $\mathcal{O}^\pm$, and they are pairwise nonisomorphic

▶ • Let $V \in \mathcal{O}^+$ be irreducible. Condition b) in Def 2 guarantees that $\exists d \in \mathbb{C}$ s.t. $V[d] \neq 0$ but $V[d+\mathbb{Z}_{>0}]=0$. Then $\mathfrak{n}_+(V[d])=0$, $\mathfrak{h}(V[d]) \subseteq V[d]$. But $\dim(V[d])<\infty$ and $\mathfrak{h}$ acts on $V[d]$ by pairwise commuting operators, hence, they have a common eigenvector $v \in V[d] \setminus \{0\}$ (as in Prop 1).

<u>So</u>: $\mathfrak{n}_+(v)=0$, $h(v)=\lambda(h)\cdot v$ $\forall h \in \mathfrak{h}$ with some $\lambda \in \mathfrak{h}^*$.

<u>This implies</u>: $\exists$ nonzero $\mathfrak{g}$-homom. $\phi: M^+_\lambda \longrightarrow V$, $v^+_\lambda \longmapsto v$

But $0 \neq \mathrm{Im}(\phi) \subseteq V$, V-irreducible $\Rightarrow \mathrm{Im}\,\phi = V$

Then $\mathrm{Ker}(\phi)$ is a proper graded submodule of $M^+_\lambda \Rightarrow \mathrm{Ker}(\phi) \subseteq J^+_\lambda$, hence, the map ϕ factors through $M^+_\lambda \twoheadrightarrow M^+_\lambda / J^+_\lambda = L^+_\lambda$, i.e. $V \twoheadrightarrow L^+_\lambda \Rightarrow V \simeq L^+_\lambda$.

Both graded irreducible

• The proof for $\mathcal{O}^-$ is analogous

• Finally $L^+_{\lambda_1} \not\simeq L^+_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$ as L^+_λ has a unique (up to scaling) singular vector ▥