Lecture #6

Last time: Categories $\mathcal{O}^{\pm}$

Today: - Continuation (interplay $\mathcal{O}^+$ vs $\mathcal{O}^-$), restricted dual, contravariant form
- Construction of Vir $\curvearrowright$ Fock modules

• Let's start with a general useful notion:

Def 1: A highest weight (resp. lowest weight) $\mathfrak{g}$-module is a $\mathbb{Z}$-graded $\mathfrak{g}$-module which is a quotient of M^+_λ (resp. M^-_λ)

Equivalently, V is a highest weight λ $\mathfrak{g}$-module if

$$\boxed{\exists v \in V \text{ s.t. } \mathfrak{n}_+(v)=0,\ h(v)=\lambda(h)\cdot v\ \forall h\in\mathfrak{h},\ V=\mathcal{U}(\mathfrak{g})v}$$

Thus, any highest (resp. lowest) weight λ module of $\mathfrak{g}$ fits into

$$\boxed{M^{\pm}_\lambda \twoheadrightarrow V \twoheadrightarrow L^{\pm}_\lambda}$$

As $\mathrm{Ker}(M^{\pm}_{\pm\lambda} \twoheadrightarrow L^{\pm}_{\pm\lambda}) = J^{\pm}_{\pm\lambda}$ is the kernel of $(\ ,\ )_\lambda$ introduced earlier, we get:

Corollary 1: For any highest weight λ module V^+_λ and lowest weight $-\lambda$ module $V^-_{-\lambda}$, there is a $\mathfrak{g}$-inv. form $(\ ,\ ): V^+_\lambda \times V^-_{-\lambda} \longrightarrow \mathbb{C}$

• Later on we shall use the notion of characters:

Def 2: For $M\in\mathcal{O}^+$ define its character ch_M via

$$\boxed{ch_M(q,x) = \sum_{d\in\mathbb{C}} q^{-d}\, \mathrm{Tr}_{M[d]}(e^x)}$$ – a function of $x\in\mathfrak{h}$, q-formal parameter

Remark: a) For $x=0$, we just get $\sum_d q^{-d}\dim(M[d])$ encoding all dimensions

b) When $\mathfrak{g}$-simple f.dim Lie algebra, setting $q=1$ recovers usual character

c) If $\exists L_0\in\mathfrak{h}=\mathfrak{g}_0$ s.t. $L_0|_{M[d]} = d\cdot \mathrm{Id}$, then we can formally rewrite

$$\sum_{d\in\mathbb{C}} q^{-d}\,\mathrm{Tr}_{M[d]}(e^x) = \sum_{d\in\mathbb{C}} \mathrm{Tr}_{M[d]}(e^{-\log(q)\cdot L_0}\cdot e^x) = \mathrm{Tr}_M(e^{x-\log(q)\cdot L_0})$$

$\boxed{\text{Exercise}}$ Prove $ch_{M^+_\lambda}(q,x) = e^{\lambda(x)}\cdot q^{\deg(v^+_\lambda)} \Big/ \prod_{k>0} \det_{\mathfrak{g}_{-k}}(1-q^k e^{\mathrm{ad}(x)})$

Hint: Use Molien's formula from linear algebra: $\sum_{n\geq 0} q^n\,\mathrm{Tr}_{S^n V}(S^n A) = \frac{1}{\det(1-qA)}$

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- Restricted duals

Recall that if $\mathfrak{g} \curvearrowright V$ then $\mathfrak{g} \curvearrowright V^* = \{\text{linear maps } V \to \Bbbk\}$ via

$$(x\varphi)(v) = -\varphi(xv) \quad \forall \varphi \in V^*, v \in V, x \in \mathfrak{g}$$

However, when V is an ∞-dim vector space, V^* is not a nice object to work with. However, if $V = \bigoplus_{d \in \mathbb{C}} V[d]$ is $\mathbb{C}$-graded with $\dim(V[d]) < \infty$, there is a much better behaved vector space:

Def 3: The restricted dual $V^\vee \subseteq V^*$ is defined as $V^\vee := \bigoplus_{d \in \mathbb{C}} (V[d])^*$

If we want to make $V^\vee$ into a $\mathbb{C}$-graded v. space such that $V^{\vee\vee} \simeq V$, there are two natural options for grading on $V^\vee$: placing $(V[d])^*$ in degrees $\pm d$. However, if V was a graded $\mathfrak{g}$-module, only one choice will work:

Lemma 1: If $V = \bigoplus_{d \in \mathbb{C}} V[d]$ is a $\mathbb{C}$-graded $\mathfrak{g}$-module with $\dim(V[d]) < \infty \ \forall d$, then the restricted dual is $V^\vee = \bigoplus_{d \in \mathbb{C}} V^\vee[d]$ with $V^\vee[d] = (V[-d])^*$

Proof is an easy exercise (done in class)

Note that a $\mathbb{C}$-graded $\mathfrak{g}$-module map $\varphi: V \to W$ induces $\varphi^*: W^\vee \to V^\vee$.
The above allows to relate $\mathcal{O}^+$ & $\mathcal{O}^-$:

Proposition 1: There are two mutually inverse antiequivalences $\mathcal{O}^+ \xrightarrow{\vee} \mathcal{O}^-$ and $\mathcal{O}^- \xrightarrow{\vee} \mathcal{O}^+$ defined by taking restricted dual

▸ Follows from above and $-\{d, d-1, d-2, \dots\} = \{-d, -d+1, -d+2, \dots\}$ ∎

As $(\ ,\ )_\lambda : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ was of degree zero, and $M^\pm_{\pm\lambda} \in \mathcal{O}^\pm$, we can interpret it as a graded $\mathfrak{g}$-module morphism $M^+_\lambda \to (M^-_{-\lambda})^\vee$ whose kernel $= J^+_\lambda$ (exercise!) Moreover, this factors through

$$\boxed{M^+_\lambda \twoheadrightarrow L^+_\lambda \xrightarrow{\sim} (L^-_{-\lambda})^\vee \hookrightarrow (M^-_{-\lambda})^\vee}$$

In particular, $(M^-_{-\lambda})^\vee \simeq M^+_\lambda$ generically in λ (i.e. when M^+_λ is irreducible)

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• Involutions & Shapovalov form

One can often reduce the discussion of $(,)_\lambda$ from last week just to so-called contravariant form on a single Verma module M^+_λ.

Setup: Assume the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ is equipped with an involutive automorphism $\omega: \mathfrak{g} \to \mathfrak{g}$ s.t. $\omega(\mathfrak{g}_n) = \mathfrak{g}_{-n}$, $\omega|_{\mathfrak{g}_0} = -\mathrm{Id}_{\mathfrak{g}_0}$

Examples: 1) $\mathfrak{g} = \mathcal{A}$, $\omega: \mathcal{A} \circlearrowleft$ given by $a_n \mapsto -a_{-n}$, $K \mapsto -K$

2) $\mathfrak{g} = Vir$, ω given by $L_n \mapsto -L_{-n}$, $C \mapsto -C$

3) $\mathfrak{g}$ - simple fin. dim, $\omega: e_i \mapsto f_i$, $f_i \mapsto e_i$, $h_i \mapsto -h_i$

4) $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$, $\omega: xt^k \mapsto \omega(x)t^{-k}$, $K \mapsto -K$

$\boxed{\text{Exercise}}$: Verify in all above examples that ω satisfies desired properties.

In the above setup, given any action $\mathfrak{g} \overset{\rho}{\curvearrowright} V$ we can twist it by ω, i.e. construct $\mathfrak{g} \overset{\rho^\omega}{\curvearrowright} V$ given by $\rho^\omega(x)(v) = \rho(\omega(x))(v)$. However, if $\mathfrak{g}$ was $\mathbb{Z}$-graded, V - $\mathbb{C}$-graded, then we need to reverse grading on V in ρ^ω. Hence, we get equivalences of categories $\mathcal{O}^+ \xrightarrow{\omega} \mathcal{O}^-$ and $\mathcal{O}^- \xrightarrow{\omega} \mathcal{O}^+$. Finally, composing with the restricted dual functor, we obtain:

$$\boxed{c: \mathcal{O}^+ \xrightarrow{\vee} \mathcal{O}^- \xrightarrow{\omega} \mathcal{O}^+, \quad M \mapsto M^c = \text{"contragredient" module}}$$

$\boxed{\text{Lemma 2 (exercise): The map } x \otimes v^+_\lambda \mapsto \omega(x) \otimes v^-_{-\lambda} \text{ induces } M^+_\lambda \xrightarrow{\sim} (M^-_{-\lambda})^\omega}$

note that we use $\omega|_{\mathfrak{g}_0} = -\mathrm{Id}$ in this result!

Thus, a $\mathfrak{g}$-invariant form $(,)_\lambda: M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ can be viewed as $M^+_\lambda \times (M^+_\lambda)^\omega \to \mathbb{C}$. But $(M^+_\lambda)^\omega$ coincides with M^+_λ as a vector space. Hence, we can interpret $(,)_\lambda$ as contravariant form (also called Shapovalov form)

$$\boxed{M^+_\lambda \times M^+_\lambda \xrightarrow{(,)} \mathbb{C}, \quad \text{s.t. } (xv, w) + (v, \omega(x)w) = 0, \quad (v^+_\lambda, v^+_\lambda) = 1}$$

As before, it gives $M^+_\lambda \to (M^+_\lambda)^c$ factoring through $M^+_\lambda \twoheadrightarrow L^+_\lambda \xrightarrow{\sim} (L^+_\lambda)^c \hookrightarrow (M^+_\lambda)^c$.

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Similarly to Corollary 1 on p. 1, we get:

$\boxed{\text{Corollary 2: Any highest weight module carries a Shapovalov form}}$

Moreover, since $\omega^2 = \mathrm{Id}_{\mathfrak{g}}$, the transposed form satisfies the same defining properties. Hence:

$\boxed{\text{Corollary 3: The Shapovalov form is symmetric}}$

• Virasoro action on Fock modules

For the next few weeks we shall focus on Vir-module constructions (this follows [Kac - Raina], but we'll use slightly different arguments)

Def 4: A *derivation* of a Lie algebra $\mathfrak{a}$ is a linear map $D: \mathfrak{a} \to \mathfrak{a}$ s.t.

$$D([a,b]) = [D(a), b] + [a, D(b)] \quad \forall a, b \in \mathfrak{a}$$

Now in the context when $\mathfrak{g} \curvearrowright \mathfrak{a}$ via derivations one has a semidirect product:

Def 5: If a Lie algebra $\mathfrak{g}$ acts on a Lie algebra $\mathfrak{a}$ by derivations, then the *semidirect product* $\mathfrak{g} \ltimes \mathfrak{a}$ is a Lie algebra whose underlying vector space is $\mathfrak{g} \oplus \mathfrak{a}$ and the Lie bracket is:

$$\boxed{[(x,\alpha), (y,\beta)] = ([x,y], [\alpha,\beta] + x(\beta) - y(\alpha)) \quad \forall x, y \in \mathfrak{g},\ \alpha, \beta \in \mathfrak{a}}$$

Exercise: Check the above endows $\mathfrak{g} \oplus \mathfrak{a}$ with a Lie algebra structure

Remark: a) $\mathfrak{g}$ and $\mathfrak{a}$, naturally embedded into $\mathfrak{g} \ltimes \mathfrak{a}$, are Lie subalg-s.

b) $[(x,0), (0,\beta)] = (0, x(\beta)) \quad \forall x \in \mathfrak{g},\ \beta \in \mathfrak{a}$

c) if $\mathfrak{g} \curvearrowright \mathfrak{a}$ by zero operators, we recover $\mathfrak{g} \oplus \mathfrak{a}$ = direct sum.

Basic Question: If $\mathfrak{a} \curvearrowright V$, can we extend it to $\mathfrak{g} \ltimes \mathfrak{a} \curvearrowright V$.

Today & next week we shall address it for $\mathfrak{a} = \mathcal{A}$, $\mathfrak{g} = W$ or Vir, $V = F_\mu$.

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To this end, we start with the following basic observation:

Lemma 3: The formula $f\partial_t(g,\alpha) := (fg', 0)$ $\forall f,g \in \mathbb{C}[t,t^{-1}]$, $\alpha \in \mathbb{C}$ provides an action of the Witt algebra W on the Heisenberg algebra $\mathcal{A}$ by derivations

▸ • The fact it gives an action follows from

$$(fg' - gf')h' = f(gh')' - g(fh')'$$

• The fact the above operators are derivations of $\mathcal{A}$ follows from

$$h(fg')' + fh'g' = (hfg')' \quad \& \quad \mathrm{Res}_{t=0}(hfg')'dt = 0$$

∎

Let's write down this action explicitly in the basis $\{L_n = -t^{n+1}\partial_t\}$ of W and $\{a_m = t^m\} \cup \{K\}$ of $\mathcal{A}$: $-t^{n+1}\partial_t(t^m) = -m \cdot t^{n+m}$, i.e.

$$\boxed{[L_n, a_m] = -m \cdot a_{n+m}, \quad [L_n, K] = 0 \quad \forall n, m}$$

So: If $\mathcal{A} \curvearrowright V$, then extending this to $W \ltimes \mathcal{A} \curvearrowright V$ amounts to constructing linear maps $L_n : V \to V$ such that:

1) $[L_n, a_m] = -m \cdot a_{n+m}$ $\forall m \in \mathbb{Z}$, and $[L_n, K] = 0$

2) $[L_n, L_m] = (n-m)L_{n+m}$

Question: Can we find such $\{L_n\}$ satisfying 1) & 2) for $V = F_\mu$ = Fock module?

The answer to 1) is "Yes" (see next lemma), but to 2) is "No", which will be easily fixed next time by replacing W with Vir.

Lemma 4: For every $n \in \mathbb{Z}$, there is a unique (up to adding const. Id) operator

$$L_n : F_\mu \to F_\mu \text{ s.t. } [L_n, a_m] = -m\, a_{n+m} \ \forall m \in \mathbb{Z}$$

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Proof of Lemma 4

• Uniqueness is easy. Indeed, if $L_n^{(1)}, L_n^{(2)}$ are two such operators, then $L_n^{(1)} - L_n^{(2)} : F_\mu \to F_\mu$ commutes with $\mathcal{A}$-action. But F_μ is an irreducible $\mathcal{A}$-module of countable dimension, hence $L_n^{(1)} - L_n^{(2)} = \text{const}\cdot \mathrm{Id}_{F_\mu}$ by Dixmier's lemma

• To construct such operator explicitly, let's introduce a very important notion of a normal ordering:

Def 6: For $n, m \in \mathbb{Z}$, the normally ordered product $:a_m a_n: \in \mathcal{U}(\mathcal{A})$ is

$$\boxed{:a_m a_n: = \begin{cases} a_m a_n & \text{if } m \le n \\ a_n a_m & \text{if } m > n \end{cases}}$$

Remark: a) $:a_m a_n: = :a_n a_m:$

b) $[x, :a_m a_n:] = [x, a_m a_n] \quad \forall x \in \mathcal{U}(\mathcal{A})$

With this notion, let's define $L_n : F_\mu \to F_\mu$ via

$$\boxed{L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} :a_{-j}a_{j+n}:}$$

← though an infinite sum, its action on F_μ is well-defined

Rmk: a) $L_0 = \frac{\mu^2}{2} + \sum_{j>0} a_{-j}a_j = \frac{\mu^2}{2} + E$ (E - Euler field)

b) For $n \neq 0$: $L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}$

Then:

$$[L_n, a_m] = \frac{1}{2}\sum_j [:a_{-j}a_{j+n}:, a_m] = \frac{1}{2}\sum_j [a_{-j}a_{j+n}, a_m] = \frac{1}{2}\sum_j \left([a_{-j}, a_m]a_{j+n} + a_{-j}[a_{j+n}, a_m]\right)$$

$$= \frac{1}{2}\left(-mK\cdot a_{m+n} + a_{m+n}(-mK)\right) \underset{K|_{F_\mu} = \mathrm{Id}}{=\!=} -m\,a_{m+n}.$$

Thus, the above L_n satisfies the desired commutation rel-s ∎