Lecture #7

Last time: 1) $^\vee$, ω, Shapovalov form, semidirect product

2) Vir-action on Fock spaces

Today: Continue 2) + generalize.

- Finish the verification of $[L_n, a_m] = -m a_{n+m}$ for

$$L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} :a_{-j}\, a_{j+n}:$$

This gives a positive answer to Question 1) from Lecture 6.

- However, we shall now show that the commutation relation b/w the operators $\{L_n\}_{n\in\mathbb{Z}}$ is not that of Witt algebra, but rather of Virasoro:

> **Proposition 1:** The operators $L_n : F_\mu \to F_\mu$ defined above satisfy:
> $$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\,\mathrm{Id}_{F_\mu}$$

Note that $\mathrm{Vir} \twoheadrightarrow W$ and so we can also consider $\mathrm{Vir} \ltimes \mathcal{A}$. Then, this result (together with last lemma from Lecture 6) imply that $\mathcal{A} \curvearrowright F_\mu$ can be extended to $\mathrm{Vir}\ltimes\mathcal{A} \curvearrowright F_\mu$. This gives an action of Virasoro on Fock modules F_μ, $C \mapsto \mathrm{Id}$

Remark: In particular, our normalization $\frac{n^3-n}{12}$ is precisely made to let $C \mapsto \mathrm{Id}_{F_\mu}$.

Proof of Proposition 1

- One direct proof, based on above $[L_n, a_m] = -m\, a_{n+m}$, is given in [Kac - Raina, Proposition 2.3]
- Let's present a different argument.

Step 1: Verify that $[L_n, L_m] - (n-m)L_{n+m}$ commutes with all $\{a_k\}$:

$$\big[[L_n, L_m] - (n-m)L_{n+m}, a_k\big] = \big[[L_n, a_k], L_m\big] + [L_n, [L_m, a_k]] - [L_{n+m}, a_k]\cdot(n-m)$$
$$= [-k\, a_{n+k}, L_m] + [L_n, -k\, a_{m+k}] + k(n-m)a_{n+m+k} = (-k(n+k) + k(m+k) + k(n-m))a_{n+m+k} = 0$$

So: by Dixmier's lemma $[L_n, L_m] - (n-m)L_{n+m} = \mathrm{const}\cdot\mathrm{Id}_{F_\mu}$.

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(Continuation of Proof)

Step 2: Since F_μ is $\mathbb{Z}$-graded and $\deg(L_n)=n$, we see that $[L_n,L_m]-(n-m)L_{n+m}$ is a degree $n+m$ operator. As $\deg(\mathrm{Id})=0$, we get above difference vanishes for $n+m\neq 0$.

Step 3: Let γ_n denote the constants s.t. $[L_n, L_{-n}]-2n\cdot L_0=\gamma_n\cdot \mathrm{Id}_{F_\mu}$.
It then remains to show $\gamma_n=\frac{n^3-n}{12}$

But considering the map $W\times W\to\mathbb{C}$ given by $(L_n,L_m)\mapsto \gamma_n\delta_{n,-m}$ we see it's tautologically is a 2-cocycle. Hence, by Theorem 1 of Lecture 1, there are $c\in\mathbb{C}$, $\xi\in W^*$ s.t.

$$\boxed{\gamma_n\delta_{n,-m}=c\cdot\frac{n^3-n}{6}\delta_{n,-m}+\xi([L_n,L_m])\quad \forall n,m}$$

so that $\gamma_n=c\cdot\frac{n^3-n}{6}+2n\cdot\xi(L_0)$. We shall now evaluate $c,\xi(L_0)$:

(1) $L_0(1)=\frac{\mu^2}{2}\cdot 1$; $[L_1,L_{-1}](1)=L_1L_{-1}(1)=L_1(\mu x_1)=\mu^2\cdot 1$

$$\Rightarrow \underbrace{([L_1,L_{-1}]-2L_0)}_{\gamma_1\cdot\mathrm{Id}}(1)=0 \Rightarrow \gamma_1=0\Rightarrow c\cdot 0+2\xi(L_0)=0\Rightarrow \boxed{\xi(L_0)=0}$$

(2) $[L_2,L_{-2}](1)=L_2L_{-2}(1)=L_2(\mu x_2+\frac{x_1^2}{2})=2\mu^2+\frac{1}{2}$

$$\Rightarrow \underbrace{([L_2,L_{-2}]-4L_0)}_{\gamma_2\cdot\mathrm{Id}}(1)=\frac{1}{2}\Rightarrow\gamma_2=\frac{1}{2}\Rightarrow \frac{1}{2}=c\cdot\frac{2^3-2}{6}+4\underbrace{\xi(L_0)}_{=0}\Rightarrow\boxed{c=\frac{1}{2}}$$

This implies $\gamma_n=\frac{n^3-n}{12}$ as needed, completing the proof ∎

The above construction admits a 1-parameter generalization (see Kac-Raina Sect 3.4)

Proposition 2: Fix $\lambda,\mu\in\mathbb{C}$. The following operators $\tilde{L}_n: F_\mu\to F_\mu$

$$\begin{cases}\tilde{L}_n=\frac{1}{2}\sum_{j\in\mathbb{Z}}:a_{-j}a_{j+n}:+i\lambda n a_n=\frac{1}{2}\sum_{j\in\mathbb{Z}}a_{-j}a_{j+n}+i\lambda n a_n, & n\neq 0\\ \tilde{L}_0=\frac{\lambda^2+\mu^2}{2}+\sum_{j>0}a_{-j}a_j\end{cases}$$

define an action $Vir \curvearrowright F_\mu$ with central charge $c=1+12\lambda^2$. Moreover, $[\tilde{L}_n,a_m]=-ma_{n+m}+i\lambda m^2\delta_{m,-n}$ $\forall n,m\in\mathbb{Z}$. (For $\lambda=0$, we recover previous action)

Exercise: Prove this result!

We shall now quickly introduce the notion of Hermitian/unitary modules.

Setup: $\mathfrak{g}$ – $\mathbb{C}$ – Lie algebra
$\dagger: \mathfrak{g} \to \mathfrak{g}$ – antilinear anti-involution, i.e. $(a^\dagger)^\dagger = a$, $(\lambda a)^\dagger = \bar{\lambda} \cdot a^\dagger$, $[a,b]^\dagger = -[a^\dagger, b^\dagger]$

Such data of $\dagger$ is called a "real structure" on $\mathfrak{g}$.

Exercise: Verify that $\mathfrak{g}_{\mathbb{R}} := \{a \in \mathfrak{g} \mid a^\dagger = -a\}$ is an $\mathbb{R}$-Lie algebra, s.t. $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$.

In our context when $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra$/\mathbb{C}$, we we shall assume $(\mathfrak{g}_k)^\dagger = \mathfrak{g}_{-k}$ $\forall k$. In particular, $\dagger: \mathfrak{g}_0 \to \mathfrak{g}_0$, and can take $\mathfrak{g}_{0,\mathbb{R}}$ as above.

Def 1: Let V be a module over $\mathbb{C}$-Lie algebra with a real structure $\dagger$.
1) V is called Hermitian if it's equipped with a nondegenerate Hermitian form as $\mathbb{C}$-vector space, s.t. $(av, w) = (v, a^\dagger w)$ $\forall v, w \in V$, $a \in \mathfrak{g}$
2) V is an 1) is called unitary if this form is positive definite.

Exercise: For any $\lambda \in \mathfrak{g}_{0,\mathbb{R}}^*$, show that M_λ^+ carries a unique Hermitian form $(\ ,\ )$ s.t. $(v_\lambda^+, v_\lambda^+) = 1$. It descends to nondegenerate Hermitian form on L_λ^+.

Examples:
1) $\mathfrak{g} = \mathcal{A}$, $a_k^\dagger = a_{-k}$, $K^\dagger = K$
2) $\mathfrak{g} = Vir$, $L_k^\dagger = L_{-k}$, $C^\dagger = C$
3) $\mathfrak{g}$ - simple, $e_i^\dagger = f_i$, $f_i^\dagger = e_i$, $h_i^\dagger = h_i$
4) $\hat{\mathfrak{g}}$, $(at^k)^\dagger = a^\dagger t^{-k}$, $K^\dagger = K$.

The reason why the above notion is useful is explained in:

Lemma 1: If V is Hermitian $\mathfrak{g}$-module, then any $\mathfrak{g}$-submodule W of V admits a complementary submodule, i.e. $V = W \oplus W'$

► Take W' to be orthogonal complement of W w.r.t. form ☐

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Very similarly to the previous result, we get:

Lemma 2: Any highest weight Hermitian $\mathfrak{g}$-module V is irreducible

► Let λ be the highest weight of the given V, so that $V \overset{\phi}{\twoheadrightarrow} L_\lambda^+$.
Then $W := \mathrm{Ker}(\phi)$ is a $\mathfrak{g}$-submodule of V, and as in previous proof:
$V = W \oplus W^\perp$, $W^\perp$ = orthogonal complement. But then $W^\perp \tilde{\twoheadrightarrow} L_\lambda^+$, so that
$V \simeq L_\lambda^+ \oplus W$. But $\dim(\text{top degree}) = 1$, spanned by $v_\lambda^+ \in L_\lambda^+ \Rightarrow$ it generates only L_λ^+, not $W \Rightarrow$ ∎

Applying this result iteratively, we get:

Corollary 1: Any Hermitian module in $\mathcal{O}^+$ is a direct sum of irreducibles

Let's now get back to today's setup of Vir & $\mathcal{A}$ acting on F_μ.

Lemma 3: For $\mu \in \mathbb{R}$, $\exists!$ Hermitian form $(\,,\,)$ on F_μ w.r.t. $\mathcal{A}$-action such that $(1,1)=1$. Moreover, it's actually unitary.

► Recall that $a_k^+ = a_{-k}$, and $a_{-j} \mapsto x_j$, $a_j \mapsto j\partial_{x_j}$ for $j>0$.
It's then easy to see that $(x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots, x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots)$ vanishes unless $n_i = m_i$ $\forall i$, and equals $\prod n_i! \cdot \prod i^{n_i}$ if $m_i = n_i$ $\forall i$. Here, we use $(k\partial_{x_k})^n (x_k^n) = k^n \cdot n!$ It's also easy to see this form satisfies $(a_k v, w) = (v, a_{-k} w)$. Obviously the form is positive definite ∎

Viewing F_μ as $\mathcal{A}$-module through Lemmas 1-2 do not give anything useful as it's irreducible. However, we can also view F_μ as a module over Vir, with the action specified in Proposition 2 (depending on $\lambda \in \mathbb{R}$). The point is that it's also unitary in that case

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Lemma 4: If $\lambda, \mu \in \mathbb{R}$, then F_μ is Hermitian w.r.t. Vir-action of Prop 2. Moreover, it's unitary!

▷ $\tilde{L}_n^+ = \frac{1}{2}\sum_{j\in\mathbb{Z}} (a_{-j}a_{n+j})^+ + (i\lambda n a_n)^+ = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-n-j}a_j - i\lambda n a_{-n} = \tilde{L}_{-n}$ (it's here that we use $\lambda \in \mathbb{R}$)

$\tilde{L}_0^+ = \frac{\lambda^2+\mu^2}{2} + \sum_{j>0} (a_{-j}a_j)^+ = \tilde{L}_0$

∎

We conclude by noting that $1 \in F_\mu$ is singular w.r.t. Vir-action of Prop 2, with highest weight $h = \frac{\lambda^2+\mu^2}{2}$, $c = 1+12\lambda^2$, so that we get

$$\boxed{\text{Vir-module homomorphism } M^+_{h=\frac{\lambda^2+\mu^2}{2},\, c=1+12\lambda^2} \xrightarrow{\phi} F_\mu}$$

This immediately implies:

Corollary 2: The irreducible highest weight Vir-module $L_{h,c}$ is unitary if $c \geq 1$ and $h \geq \frac{c-1}{24}$

▷ $\mathrm{Im}(\phi)$ is a highest weight submodule of unitary module F_μ
$\Rightarrow \mathrm{Im}(\phi)$ is unitary and in fact $\mathrm{Im}(\phi) \simeq L^+_{h,c}$ by Lemma 2 ∎

We also note that both $M^+_{h,c}$ & F_μ are $\mathbb{Z}$-graded vector spaces, with the same dimensions ($\dim(\text{degree-}n \text{ part}) = p(n) =$ number of partitions of n)

Since $M^+_{h,c}$ is irreducible for "generic" h, c, we obtain:

Corollary 3: The above homomorphism $\phi: M^+_{h,c} \to F_\mu$ is generically isomorphism

Exercise: Show that $x_1 \in F_0$ is singular w.r.t. Vir-action with $\lambda=\mu=0$. Thus ϕ cannot be always an isomorphism.