Lecture #8

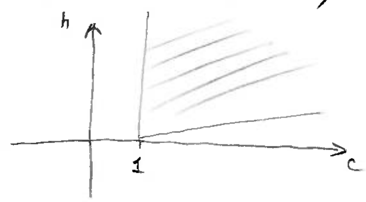
Last time: 1) $Vir \curvearrowright F_\mu$ with central charge $c = 1+12\lambda^2$

2) Unitary & Hermitian modules

3) Unitarity of above $Vir \curvearrowright F_\mu$ and $Vir \curvearrowright L_{h,c}$ with $h = \frac{\lambda^2+\mu^2}{2}$, $c = 1+12\lambda^2$ $(\lambda, \mu \geq 0)$

Today: Further discussion; Construction of $c=\frac{1}{2}$ unitary Vir-modules; quantum fields.

- Start by evoking Lemma 2, Corollary 3 from Lecture 7 notes
- Note that the region of $(h,c) \in \mathbb{R}^2_{\geq 0}$ s.t. $\exists\, \lambda, \mu \in \mathbb{R}$ so that $h = \frac{\lambda^2+\mu^2}{2}$, $c = 1+12\lambda^2$ can be described as $\{(h,c) \mid c \geq 1,\ h \geq \frac{c-1}{24}\}$ and is depicted by



The following result is obvious:

$\boxed{\text{Lemma 1: A Vir-module } L^+_{h,c} \text{ is unitary only if } h, c \geq 0}$

► For any $n \geq 0$: $\underbrace{\langle L_{-n} v^+_\lambda, L_{-n} v^+_\lambda \rangle}_{\langle L_n L_{-n} v^+_\lambda,\, v^+_\lambda \rangle} \in \mathbb{R}_{\geq 0}$

$\langle L_n L_{-n} v^+_\lambda, v^+_\lambda \rangle = 2n \cdot h + \frac{n^3-n}{12} \cdot c$

- for $n=1 \Rightarrow h \in \mathbb{R}_{\geq 0}$
- for $n \gg 1$, the leading term is $\frac{n^3-n}{12} c$, hence, $c \in \mathbb{R}_{\geq 0}$ ∎

- Note that if V_1, V_2 are unitary $\mathfrak{g}$-modules, then so is $V_1 \otimes V_2$, with the Hermitian form defined termwise, i.e.

$$\boxed{(v_1 \otimes w_1, v_2 \otimes w_2) = (v_1, v_2) \cdot (w_1, w_2) \quad \forall v_1, v_2 \in V_1,\ w_1, w_2 \in V_2}$$

In particular, given $\{(h_i, c_i)\}_{i=1}^N$ s.t. $c_i \geq 1$, $h_i \geq \frac{c_i - 1}{24}$, we see that $L^+_{h_1,c_1} \otimes \ldots \otimes L^+_{h_N,c_N}$ is unitary. Thus, the highest weight module generated by $v^+_{h_1,c_1} \otimes \ldots \otimes v^+_{h_N,c_N}$ is also unitary, hence by Lemma 2 of Lecture 7, it's irreducible $L^+_{h_1+\ldots+h_N,\, c_1+\ldots+c_N}$.

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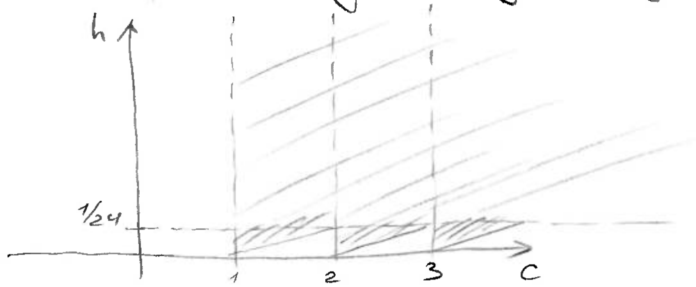
The above discussion implies:

Corollary 1: If $c \geq m$ & $h \geq \frac{c-m}{24}$ for some $m \in \mathbb{Z}_{>0}$, then $L^+_{h,c}$ - unitary

▶ Apply the previous setup to $(h_1, c_1) = \ldots = (h_{m-1}, c_{m-1}) = (0,1)$, $(h_m, c_m) = \left(\frac{c-m}{24}, c-m+1\right)$ ■

This can be depicted by the following region (see [Kac-Raina, Fig 3.1]):



A big result which we can only state now is:

Theorem 1: $L^+_{h,c}$ is unitary for any $c \geq 1$ & $h \geq 0$

Clearly this does not classify all unitary $L^+_{h,c}$, e.g. $L^+_{0,0} = \underset{1\text{-dim}}{\text{trivial}}$ is unit!

We shall now present a similar construction resulting in unitary Vir-modules $L^+_{h,c=\frac{1}{2}}$

- "Super-analogue" of previous construction (see [Kac-Raina, §3.4])

Def 1: Fix $\delta \in \{0, \frac{1}{2}\}$ and let C_δ be the algebra generated by $\{\psi_m\}_{m \in \delta + \mathbb{Z}}$ (called fermions) subject to

$$\boxed{[\psi_m, \psi_n]_+ := \psi_m \psi_n + \psi_n \psi_m = \delta_{m,-n}}$$

↑ (super-analogue of $a_m a_n - a_n a_m = m\delta_{m,-n}$)

Common terminology: $\delta = \frac{1}{2} \rightsquigarrow$ "Neveu-Schwarz" sector

$\delta = 0 \rightsquigarrow$ "Ramond" sector

Consider the space of polynomials in anti-commuting variables

$$\boxed{V_\delta = \Lambda\{\xi_n \mid n \in \delta + \mathbb{Z}_{\geq 0}\}, \quad \xi_n \xi_m = -\xi_m \xi_n}$$

↑ super-analogue of Fock space $\mathbb{C}[x_1, x_2, \ldots]$

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Lemma 2: The algebra C_δ acts on V_δ via

$$\psi_{-n} \mapsto \xi_n = \text{left multiplication by } \xi_n$$

$$\psi_n \mapsto \partial/\partial\xi_n \text{ defined via } \partial/\partial\xi_n(\xi_{i_1}\wedge\ldots\wedge\xi_{i_k}) = \begin{cases} 0 & \text{if } n\notin\{i_1,\ldots,i_k\} \\ (-1)^{r-1}\xi_{i_1}\wedge\ldots\wedge\xi_{i_{r-1}}\wedge\xi_{i_{r+1}}\wedge\ldots\wedge\xi_{i_k} & \text{if } n=i_r \end{cases} \quad i_1<\ldots<i_k.$$

$$\psi_0 \overset{\text{if }\delta=0}{\longmapsto} \frac{1}{\sqrt{2}}(\xi_0 + \partial/\partial\xi_0)$$

► Direct check – do in class ■

Note: For $\delta=0$, we note that $\psi_0^2 = 1/2$.

We can now present a "super-analogue" of Vir-action on V_δ:

Proposition 1: For $\delta\in\{0,1/2\}$, define $L_n: V_\delta \to V_\delta$ via

$$\boxed{L_n := \delta_{n,0}\cdot\frac{1-2\delta}{16} + \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j:\psi_{-j}\psi_{j+n}:}$$

with normal ordering $:\psi_i\psi_j: = \begin{cases}\psi_i\psi_j, & i\leq j \\ -\psi_j\psi_i, & i>j\end{cases}$

the "–" sign is due to super-setup!

Then: 1) $[L_n, \psi_m] = -(\frac{n}{2}+m)\psi_{m+n}$

2) $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\cdot\frac{n^3-n}{24}$

In particular, $\{L_n\}$ give rise to Vir $\curvearrowright V_\delta$ with $c=1/2$

Exercise: Prove this result (very similar to Lecture 7).

• However, one can easily notice that V_δ are not irreducible Vir-mod.
Indeed, let's endow V_δ with $\mathbb{Z}_2$-grading so that $\deg(\xi_k) = \bar{1}\in\mathbb{Z}_2$ $\forall k$.
Then each of above L_n preserves $\mathbb{Z}_2$-grading, hence, a decomposition

$$\boxed{V_\delta = V_\delta^+ \oplus V_\delta^-, \quad \begin{matrix} V_\delta^+ = \{\text{even elements}\} \\ V_\delta^- = \{\text{odd elements}\}\end{matrix}}$$

The following big result will be proved much later:

Theorem 2: For $\delta\in\{0,1/2\}$, Vir-modules V_δ^+ and V_δ^- are irreducible

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Finally, we claim that $V_\delta^\pm$ are actually unitary Vir-modules.
To this end, we start with:

Lemma 3: Consider the Hermitian form $(\ ,\ )$ on V_δ s.t. all "ordered" monomials $\{\xi_{i_1}\wedge\xi_{i_2}\wedge\ldots\wedge\xi_{i_k} \mid i_1<i_2<\ldots<i_k\}$ form orthonormal basis.

Then: a) $(\psi_n v, w) = (v, \psi_{-n} w)$ $\forall n\in\delta+\mathbb{Z}$, $v,w\in V_\delta$

b) $(L_n v, w) = (v, L_{-n} w)$ $\forall n\in\mathbb{Z}$, $v,w\in V_\delta$

► Direct check – sketch in class ∎

Thus $V_\delta^\pm$ are indeed unitary Vir-modules.

- Note that $1\in V_0^+$ is the highest weight vector of h.wt. $(\frac{1}{16},\frac{1}{2})$

$\xi_0\in V_0^-$ —//— $(\frac{1}{16},\frac{1}{2})$

$1\in V_{1/2}^+$ —//— $(0,\frac{1}{2})$

$\xi_{1/2}\in V_{1/2}^-$ —//— $(\frac{1}{2},\frac{1}{2})$

Theorem 2 $\Rightarrow$

$$\boxed{V_0^+\simeq V_0^-\simeq L^+_{1/16,1/2}\ ,\quad V_{1/2}^+\simeq L^+_{0,1/2}\ ,\quad V_{1/2}^-\simeq L^+_{1/2,1/2}}$$

Remark: In fact, as we shall see later, these are the only unitary among $L^+_{h,1/2}$.

In fact, the following result (which we'll partially establish later) together with Theorem 1 provides a complete answer to unitarity:

Theorem 3: 1) For $0<c<1$ outside of $\{1-\frac{6}{(n+2)(n+3)} \mid n\in\mathbb{N}\}$ and any $h\in\mathbb{R}_{\geq0}$, $L^+_{h,c}$ is not unitary

2) For $c=1-\frac{6}{(n+2)(n+3)}$ with $n\in\mathbb{N}$, there are finitely many $h\in\mathbb{R}_{\geq0}$ s.t. $L^+_{h,c}$ – unitary (we'll give these h explicitly)

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- Let's compute "characters" of these 3 unitary modules $L^+_{h,1/2}$ $(h=0,\frac{1}{16},\frac{1}{2})$

i.e. $\boxed{ch(L^+_{h,1/2}) = \sum_{\lambda\in\mathbb{C}} \dim(\lambda\text{-generalized eigenspace of } -L_0)\cdot q^\lambda}$

$\underline{\underline{\delta=0}}$: $2\cdot ch(L^+_{1/16,1/2}) = ch(V_0) = q^{1/16}\prod_{k\geq 0}(1+q^k) = 2q^{1/16}\prod_{k\geq 1}(1+q^k)$

$\underline{\underline{\delta=1/2}}$: $ch(L^+_{0,1/2}) + ch(L^+_{1/2,1/2}) = ch(V_{1/2}) = \prod_{k\geq 0}(1+q^{k+1/2})$

Therefore, we obtain (see [Kac-Raina, Prop 3.8]):

Corollary 2: $ch(L^+_{1/16,1/2}) = q^{1/16}\cdot\prod_{k\geq 1}(1+q^k)$

$ch(L^+_{0,1/2}) =$ "integer part" of $\prod_{k\geq 0}(1+q^{k+1/2})$

$ch(L^+_{1/2,1/2}) =$ "half-integer part" of $\prod_{k\geq 0}(1+q^{k+1/2})$

- We conclude today's class with a quick detour into "quantum fields", just re-interpreting the earlier discussions.

Recall: a) For any vector space V, one can associate vector spaces $V[z]$, $V[[z]]$, $V[z,z^{-1}]$, $\underbrace{V((z))}_{\text{finite in } -\mathbb{Z}_{>0} \text{ direction}}$, $V[[z,z^{-1}]]$

b) If V is an algebra, the the first four of above are algebras

In physics, it's common to work with elements of $V[[z,z^{-1}]]$, called quantum fields, i.e. expressions of the form $A(z)=\sum_{n\in\mathbb{Z}} A_n z^{-n-1}$, $A_n\in V$

Note: The operator $\frac{d}{dz}$ acts on each of above 5 spaces.

Moreover, there is a unique way to truncate $A(z)$ into two semi-infinite sums $A(z)=A_+(z)+A_-(z)$ w.r.t. $\frac{d}{dz}$, namely:

$$\boxed{A_+(z)=\sum_{n\leq -1} A_n z^{-n-1}\in V[[z]], \quad A_-(z)=\sum_{n\geq 0} A_n z^{-n-1}\in z^{-1}V[[z^{-1}]]}$$

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One also crucially needs the so-called "delta function"

$$\boxed{\delta(w-z) = \sum_{n\in\mathbb{Z}} z^{-n-1} w^n}$$

A common way to write it is as $\delta(w-z) = \frac{1}{z-w} + \frac{1}{w-z}$, where

$$\boxed{\frac{1}{z-w} = \sum_{n\geq 0} z^{-n-1} w^n}$$

Key Property:

$$\boxed{\frac{1}{2\pi i}\oint_{|z|=1} \delta(w-z) f(z)dz = f(w) \qquad \forall f(z)\in V[z,z^{-1}]}$$

So: We can work with quantum fields attached to Heisenberg & Virasoro:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \qquad \& \qquad T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

and there are several contexts:

1) $a(z)\in \mathcal{A}[[z,z^{-1}]]$, $T(z)\in Vir[[z,z^{-1}]]$

2) $a(z)\in \mathcal{U}(\mathcal{A})[[z,z^{-1}]]$, $T(z)\in \mathcal{U}(Vir)[[z,z^{-1}]]$

3) $a(z)\in \mathrm{End}(V)[[z,z^{-1}]]$, $T(z)\in \mathrm{End}(V)[[z,z^{-1}]]$ for any respective module V

Example: 1) Taking $\mathcal{A}\curvearrowright F_\mu$, the defining rel-s of $\mathcal{A}$ can be encoded by

$$\boxed{[a(z),a(w)] = \sum_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1} = \partial_w \delta(w-z) = -\partial_z \delta(w-z) =: \delta'(w-z)}$$

2) Likewise, $a(z)a(w)$ can be split into "regular part" & "singular part"

$$\boxed{a(z)a(w) = :a(z)a(w): + \frac{1}{(z-w)^2}}$$

3) The action $Vir \curvearrowright F_\mu$ from Prop 1 of Lecture 7 can be described:

$$\boxed{T(z) = \frac{1}{2} :a(z)^2:}$$

4) The resulting commutation b/w L_n & a_m can be described by:

$$\boxed{[T(z), a(w)] = a(z)\delta'(w-z)}$$

5) The Virasoro rel-s among L_n can be described by

$$\boxed{[T(z), T(w)] = -T'(z)\cdot\delta(w-z) + 2T(z)\delta'(w-z) + \frac{c}{12}\delta'''(w-z)}$$

Exercise: Prove above!