Lecture #9

Today: Start discussing gl_∞, a_∞ and their representations (see Section 4 of Kac-Raina)

Def 1: gl_∞ is the Lie algebra of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ s.t. all but finitely many $a_{ij}=0$ with the usual commutator

Down-to-earth: gl_∞ has a basis $\{E_{ij} \mid i,j\in\mathbb{Z}\}$ with the Lie bracket given by

$$\boxed{[E_{ij}, E_{mn}] = \delta_{jm}\cdot E_{in} - \delta_{in}\cdot E_{mj}}$$

Similarly to how $gl_n \curvearrowright \mathbb{C}^n$, the Lie algebra gl_∞ naturally acts on the vector space $V=\bigoplus_{k\in\mathbb{Z}} \mathbb{C}\cdot v_k$ via the usual formulas

$$\boxed{E_{ij}(v_k) = \delta_{jk}\cdot v_i}$$

Moreover, this is a $\mathbb{Z}$-graded representation, where

$$\boxed{\deg(E_{ij}) = j-i, \quad \deg(v_k) = -k}$$

Remark: a) Note that $(gl_\infty)_k$ is ∞-dim $\forall k\in\mathbb{Z}$. But $(gl_\infty)_0$ is abelian!

b) The triangular decomposition $gl_\infty = n_- \oplus \mathfrak{h} \oplus n_+$ w.r.t. this grading has: n_+ = strictly upper-triangular, n_- = strictly lower-Δ.

c) For any $\lambda\in\mathfrak{h}^*$, still have Verma M^+_λ and its irreducible quot-t L^+_λ.

Similarly to how $gl_n \curvearrowright \mathbb{C}^n$ induces $gl_n \curvearrowright S^k(\mathbb{C}^n)$ $\forall k\geq 1$, $\Lambda^k(\mathbb{C}^n)$ $\forall k\leq n$, we also get gl_∞-action on any S^kV, Λ^kV $\forall k\geq 1$.

$\boxed{\text{Exercise}}$: Prove that gl_∞-modules S^kV, Λ^kV are irreducible $\forall k\geq 1$

Hint: you may use a finite dimentional counterpart of this result.

!However: None of V, S^kV, Λ^kV $(k>1)$ has highest weights and moreover none of them belongs to category $\mathcal{O}$.

Main goal: Introduce gl_∞-modules similar to Λ^kV, which will be in $\mathcal{O}$.

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Def 2: a) An <u>elementary</u> <u>semi-infinite wedge</u> ($\frac{\infty}{2}$-wedge) is a formal infinite wedge product $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots$ with indices $i_0 > i_1 > i_2 > \dots$ s.t.

$$\boxed{i_{k+1} = i_k - 1 \quad \forall k \gg 0}$$

b) The <u>semi-infinite wedge space</u> $\Lambda^{\frac{\infty}{2}} V$ is a $\mathbb{C}$ vector space with the basis given by elementary semiinfinite wedges.

Proposition 1: The usual Leibniz rule defines an action $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2}} V$

Exercise: Work out the proof of this result!

Down-to-earth, the above action is given by:

$$E_{rs}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots) = \begin{cases} 0, & \text{if } s \notin \{i_0, i_1, \dots\} \\ v_{i_0} \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}} \wedge v_r \wedge v_{i_{k+1}} \wedge \dots & \text{if } i_k = s \end{cases}$$

Warning: this vector is not an $\frac{\infty}{2}$-wedge, but is understood as 0 if $r \in \{i_0, i_1, \dots, i_{k-1}, i_{k+1}, \dots\}$ or as $\pm\frac{\infty}{2}$-wedge, where we reorder $v_i \wedge v_j = -v_j \wedge v_i$ to have indices $\downarrow$.

Def 3: For any $m \in \mathbb{Z}$, let $\Lambda^{\frac{\infty}{2}, m} V$ be the span of $\frac{\infty}{2}$-wedges with

$$\boxed{i_k + k = m \text{ for } k \gg 0}$$

Then $\Lambda^{\frac{\infty}{2}} V = \bigoplus_{m \in \mathbb{Z}} \Lambda^{\frac{\infty}{2}, m} V$ is a decomposition into $\mathfrak{gl}_\infty$-submodules.

Define.

$$\psi_m := v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots \in \Lambda^{\frac{\infty}{2}, m} V$$

$$\omega_m = (\dots \underset{m-1}{1}, \underset{m}{1}, \underset{m+1}{0}, 0, \dots) \text{ - weight of } \mathfrak{gl}_\infty$$ (i.e. $\omega_m(E_{ii}) = \begin{cases} 1, & i \le m \\ 0, & i > m \end{cases}$)

Proposition 2: a) $\mathfrak{n}_+(\psi_m) = 0$

b) $h(\psi_m) = \omega_m(h) \cdot \psi_m$

c) $\Lambda^{\frac{\infty}{2}, m} V$ is generated by ψ_m

i.e. $\Lambda^{\frac{\infty}{2}, m} V$ is a highest weight module with h.wt. vector ψ_m of h.wt ω_m

▸ a) & b) - obvious

c) Applying $E_{i_0, m}$, then $E_{i_1, m-1}$, etc. will get any basis elt of $\Lambda^{\frac{\infty}{2}, m} V$

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Proposition 3: For any $m \in \mathbb{Z}$, the module $\Lambda^{\frac{\infty}{2},m} V$ is irreducible h.weight, i.e. $\Lambda^{\frac{\infty}{2},m} V \simeq L^+_{\omega_m}$. Moreover, it's unitary w.r.t. $\dagger: \mathfrak{gl}_\infty \to \mathfrak{gl}_\infty$ given by $E_{ij}^\dagger = E_{ji}$ (i.e. $\dagger$ = transposition)

▸ While irreducibility can be checked directly, it will immediately follow from unitarity. To prove the latter, let's equip $\Lambda^{\frac{\infty}{2},m} V$ with a Hermitian form so that $\frac{\infty}{2}$-wedges form an orthonormal basis.

$\boxed{\text{Exercise}}$: Verify then $(E_{ij} w, w') = (w, E_{ji} w')$ $\forall w, w' \in \Lambda^{\frac{\infty}{2},m} V$ ■

As an immediate corollary, we get:

Corollary 1: If $\lambda = (\lambda_i)_{i \in \mathbb{Z}}$ with $\lambda_i \in \mathbb{R}$, $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$, $\lambda_i = \lambda_{i+1}$ for $|i| >> 0$, then the irreducible $\mathfrak{gl}_\infty$-module L^+_λ is unitary

▸ The proof is very similar to that of [Lecture 8, Corollary 1]. First, we note that any λ as above can be written as

$$\lambda = \beta + \omega_{i_1} + \omega_{i_2} + \_\_ + \omega_{i_N}, \quad \text{where } \beta \in \mathfrak{h}^* \text{ s.t. } \beta(E_{ii}) = \beta \ \forall i$$

By Prop 3 above, each of $L^+_{\omega_{i_k}}$ $(1 \leq k \leq N)$ is unitary. On the other hand, L^+_β is a 1dim $\mathfrak{gl}_\infty$-module with any $X \in \mathfrak{gl}_\infty$ acting via $\beta \cdot \mathrm{tr}(X)$, which is also unitary. But then: $L^+_\beta \otimes L^+_{\omega_{i_1}} \otimes \_\_ \otimes L^+_{\omega_{i_N}}$ is unitary, hence so is the highest weight $\mathfrak{gl}_\infty$-submodule generated by $v^+_\beta \otimes v^+_{\omega_{i_1}} \otimes \_\_ \otimes v^+_{\omega_{i_N}}$. However, as recalled above, unitary h.wt modules are irreducible $\Rightarrow L^+_\lambda$ is unitary as claimed. ■

The reverse turns out to be true as well, given λ_i stabilize at $|i| >> 0$ (note: this is much simpler than unitarity region of Vir-modules as discussed last time) ↓

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Proposition 4: If an irreducible $\mathfrak{gl}_\infty$-module L^+_λ is unitary, where $\lambda = (\lambda_i)$ satisfy $\lambda_i \in \mathbb{R}$ and $\lambda_i - \lambda_{i+1} = 0$ for $|i| \gg 0$, then in fact

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \quad \forall i$$

The idea is to utilize a family of $\mathfrak{sl}_2$ inside $\mathfrak{gl}_\infty$, spanned by $\{\underbrace{E_{i,i+1}}_{e_i}, \underbrace{E_{i+1,i}}_{f_i}, \underbrace{E_{ii} - E_{i+1,i+1}}_{h_i}\}$. Therefore, we start with an $\mathfrak{sl}_2$-result:

Lemma 1: An irreducible h.wt. $\mathfrak{sl}_2$-module L_μ is unitary iff $\mu \in \mathbb{Z}_{\geq 0}$

► Using the calculation from [Lecture 5, Example 1], we get:

$$(f^n v^+_\mu, f^n v^+_\mu) = (e^n f^n v^+_\mu, v^+_\mu) = n! \cdot \mu(\mu-1)\ldots(\mu-n+1).$$

Case 1: If $\mu \notin \mathbb{Z}_{\geq 0}$, then it's clear that $\mu(\mu-1) \cdot \ldots \cdot (\mu-n+1) \notin \mathbb{R}_{\geq 0}$ for some n.

Case 2: If $\mu = N \in \mathbb{Z}_{\geq 0}$, then as discussed earlier, L^+_μ has a basis $\{v^+_N, f v^+_N, \ldots, f^N v^+_N\}$ and above f-la shows L^+_μ is indeed unitary ∎

Proof of Proposition 4

► Consider the $\mathfrak{sl}_2$-subalgebra spanned by e_i, f_i, h_i above ⇝ call it $\mathfrak{sl}_2^{(i)}$. Then $\mathfrak{sl}_2^{(i)}$-submodule of L^+_λ generated by the highest weight vector $v^+_\lambda \in L^+_\lambda$ is unitary (as † on $\mathfrak{gl}_\infty$ and $\mathfrak{sl}_2^{(i)}$ are compatible) and highest weight of weight $\mu_i = \lambda_i - \lambda_{i+1}$. Hence, it must be $\simeq L^+_{\mu_i}$.

Applying Lemma 1, we get the desired $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ ∎

Remark: We note that each $\Lambda^{\frac{\infty}{2}, m} V$ is actually $\mathbb{Z}_{\leq 0}$-graded module with $\deg(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) = -\sum_{k \geq 0} (i_k + k - m)$. In particular, we see that the basis of $\Lambda^{\frac{\infty}{2}, m} V$ is parametrized by partitions = Young diagrams

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We conclude today's class with introducing a much bigger Lie algebra:

Def 4: $\overline{\mathfrak{a}}_\infty$ is the Lie algebra of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many nonzero diagonals (i.e. $\exists N$ s.t. $a_{ij}=0$ when $|i-j|>N$)

Note: Every row/column of $A=(a_{ij})\in\overline{\mathfrak{a}}_\infty$ has finitely many nonzero terms, hence, $A\cdot B$ and $A\cdot B-B\cdot A=:[A,B]$ are defined $\forall A,B\in\overline{\mathfrak{a}}_\infty$.

Warning: While $\{E_{ij}\}_{i,j\in\mathbb{Z}}$ was a basis of $\mathfrak{gl}_\infty$, the above $\overline{\mathfrak{a}}_\infty$ is actually not of countable dimension (exercise: show this!)

However, it's this bigger Lie algebra (or rather its central extension) that will be of crucial importance for us. In particular:

$$\boxed{T:=\sum_{i\in\mathbb{Z}} E_{i,i+1}\in\overline{\mathfrak{a}}_\infty \text{ acts on } V \text{ as a shift operator } v_i\mapsto v_{i-1}}$$

More generally, for any $k\in\mathbb{Z}$, $T^k:=\sum_{i\in\mathbb{Z}} E_{i,i+k}$ acts via $v_i\mapsto v_{i-k}$ $\forall i$.

Note that $\overline{\mathfrak{a}}_\infty$ is $\mathbb{Z}$-graded, similarly to $\mathfrak{gl}_\infty$, and $\deg(T^k)=k$

Here we used that $\overline{\mathfrak{a}}_\infty \curvearrowright V$ in a natural way. Note that it also induces actions $\overline{\mathfrak{a}}_\infty \curvearrowright S^k V,\ \Lambda^k V$ $\forall k\geq 1$. But what about action on $\Lambda^{\frac{\infty}{2}}V$?

$\boxed{Q}$: Can the action $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2}}V$ be extended to $\overline{\mathfrak{a}}_\infty \curvearrowright \Lambda^{\frac{\infty}{2}}V$

A: Not quite! While if $A=\sum_{i\in\mathbb{Z}} a_{i,i+k}E_{i,i+k}\in\overline{\mathfrak{a}}_\infty^k$ & $k\neq 0$, then A-action is well-defined, there is a major problem with $k=0$ case:

e.g. $\sum_{i\in\mathbb{Z}} a_{ii}E_{ii}$ should act on $\psi_0=v_0\wedge v_{-1}\wedge v_{-2}\wedge\ldots$ via a multiplication by $\sum_{i\leq 0} a_{ii}$ which is an ∞-sum.

Next time: We shall "regularize" this action, ending with a central extension $\mathfrak{a}_\infty$ on $\Lambda^{\frac{\infty}{2}}V$!