Lecture #10

<u>Last time</u>: $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V$ - irreducible h. wt. module, unitary

$\cap$

$\bar{\mathfrak{a}}_\infty$ ← ended up by this much larger Lie algebra

<u>Today</u>: Central extension $\mathfrak{a}_\infty$ of $\bar{\mathfrak{a}}_\infty$, and its action on $\Lambda^{\frac{\infty}{2},m}V$.

- But first, let me comment on the grading of $\mathfrak{gl}_\infty$-modules from Lecture 9

<u>Recall</u>: $\boxed{\deg(E_{ij}) = j-i.}$

Thus, if we want to endow V with a compatible $\mathbb{Z}$-grading, we must have $\deg(v_i) - \deg(v_j) = j-i$, so one can declare $\boxed{\deg(v_i) = -i \ \forall i.}$ This clearly makes all finite symmetric/exterior powers S^kV, Λ^kV into $\mathbb{Z}$-graded mod.

<u>However</u>: naively doing this for $\Lambda^{\frac{\infty}{2},m}V$ fails as one would end up with

$$\deg(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) = -i_0 - i_1 - i_2 - \ldots \quad \text{, an infinite sum.}$$

<u>Fix</u>: Regularize this expression via

$$\boxed{\deg(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) = \sum_{k\geq 0} -(i_k + k - m)}$$

← This sum is actually finite as $i_k + k - m = 0$ for $k \gg 0$.

In particular, we note that $\deg(v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots) = \deg(\psi_m) = 0$.

$\boxed{\text{Exercise}}$: The above formula makes $\Lambda^{\frac{\infty}{2},m}V$ into $\mathbb{Z}$-graded $\mathfrak{gl}_\infty$-module

<u>Remark</u>: We also note that elementary $\frac{\infty}{2}$-wedges that parametrize a basis of $\Lambda^{\frac{\infty}{2},m}V$ are in bijection with partitions (= Young diagrams)

$v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots \longleftrightarrow$ partition $(i_0 - m \geq i_1 + 1 - m \geq i_2 + 2 - m \geq \ldots)$

For example, $v_3 \wedge v_1 \wedge v_{-1} \wedge v_{-3} \wedge v_{-4} \wedge \ldots \in \Lambda^{\frac{\infty}{2},0}V$ tail is all 0's.

$\updownarrow$

partition $(3, 2, 1, 0, 0, \ldots)$

$\updownarrow$

Young diagram  of 6 boxes

In particular, the result that will be needed later today is:

$$\sum_{n\geq 0} \dim\left(\Lambda^{\frac{\infty}{2},m}V[-n]\right) q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\ldots} = \sum_{n\geq 0} p(n) q^n, \quad p(n) = \# \text{ partitions of } n.$$

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Let's now return to the question from the end of last class:

$\boxed{Q}$: Can the action $gl_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V$ be extended to $\overline{a}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V$?

Let's try to work it out explicitly. While we cannot appeal to any basis of $\overline{a}_\infty$, we can use the $\mathbb{Z}$-grading decomposition

$$\boxed{\overline{a}_\infty = \bigoplus_{n\in\mathbb{Z}} \overline{a}_\infty^{(n)} \,,\quad \overline{a}_\infty^{(n)} = \{(a_{ij})_{i,j} \mid a_{ij}=0 \text{ if } j-i\neq n\}}$$

matrices that are 0 outside n^{th} diagonal

Thus, any $A\in\overline{a}_\infty$ can be split into a finite sum $A=\sum_{k\in\mathbb{Z}} A_k$, $A_k\in\overline{a}_\infty^{(k)}$, and we need to specify how each A_k acts on $\Lambda^{\frac{\infty}{2},m}V$.

$\underline{k\neq 0}$: $A_k(\underbrace{v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots}_{\text{any elementary } \frac{\infty}{2}\text{-wedge in } \Lambda^{\frac{\infty}{2},m}V})$ is well-defined through Leibniz rule

Indeed, if we think of $A_k=\sum_{i\in\mathbb{Z}} a_i\cdot E_{i,i+k}$, then $E_{i,i+k}$ contributes zero if $i+k\notin\{i_0,i_1,i_2,\ldots\}$ and otherwise replaces v_{i+k} by v_i. However, as the tail of $\frac{\infty}{2}$-wedge stabilizes, if $i\ll 0$ then $i\in\{i_0,i_1,i_2,\ldots\}$ $\Rightarrow$ resulted wedge is zero. Thus, the resulting sum is finite!

$\underline{k=0}$: Here will be a problem, as explicitly we have

$$\Big(\sum a_i E_{ii}\Big)(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots) = \underbrace{(a_{i_0}+a_{i_1}+a_{i_2}+\ldots)}_{\text{infinite sum!}}\cdot v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots$$

Fix: through regularization, similarly to $\mathbb{Z}$-grading from p.1.

Our choice: modify E_{ii}-action ρ from last time via

$$\boxed{\hat{\rho}(E_{ii}) := \begin{cases} \rho(E_{ii}) & \text{if } i>0 \\ \rho(E_{ii})-1 & \text{if } i\leq 0\end{cases}} \quad \text{while} \quad \boxed{\hat{\rho}(E_{ij}) := \rho(E_{ij}) \text{ for } j\neq i}$$

Given $A=(a_{ij})\in\overline{a}_\infty$, we now consider $\hat{\rho}(A)\in \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V)$ given by

$$\boxed{\hat{\rho}(A)=\sum_{i,j} a_{ij}\,\hat{\rho}(E_{ij})}$$

$\boxed{\text{Exercise}}$: Explain why this $\hat{\rho}(A)$ is well-defined.

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Example: $\hat{\rho}(\sum a_i E_{ii})(\underbrace{v_3 \wedge v_1 \wedge v_{-1} \wedge v_{-3} \wedge v_{-4} \wedge \dots}_{\in \Lambda^{\frac{\infty}{2},0} V}) = (a_3 + a_1 - a_0 - a_{-2}) \cdot v_3 \wedge v_1 \wedge v_{-1} \wedge v_{-3} \wedge \dots$

Remark: We note that one could regularize $\hat{\rho}$ by a similar condition but rather using any other integer instead of zero, i.e. consider $\rho(E_{ii}) - \delta_{i \le R}$. But the resulting theory will be equivalent. As next week we shall be working with all $\{\Lambda^{\frac{\infty}{2},m} V\}$ at the same time, we chose the above uniform regularization. (the translation $a_{ij} \rightsquigarrow a_{i+1,j+1}$ of $\bar{\mathfrak{a}}_\infty$ makes all choices equivalent)

However: $\hat{\rho}: \bar{\mathfrak{a}}_\infty \to \mathrm{End}(\Lambda^{\frac{\infty}{2},m} V)$ is not a Lie algebra homomorphism.

To fix this, let's study the "error" of this, i.e. define

$$\boxed{\alpha(A,B) := [\hat{\rho}(A), \hat{\rho}(B)] - \hat{\rho}([A,B]) \quad \forall A, B \in \bar{\mathfrak{a}}_\infty}$$

Example: 1) $\alpha(E_{ij}, E_{k\ell}) = 0$ if $(k,\ell) \neq (j,i)$, b/c $[\hat{\rho}(A), \hat{\rho}(B)] = [\rho(A), \rho(B)]$, $\hat{\rho}(E_{st}) = \rho(E_{st})$ for $s \neq t$.

2) $\alpha(E_{ij}, E_{ji}) = [\rho(E_{ij}), \rho(E_{ji})] - \hat{\rho}(E_{ii} - E_{jj})$

$$= (\rho(E_{ii}) - \hat{\rho}(E_{ii})) - (\rho(E_{jj}) - \hat{\rho}(E_{jj})) = \delta_{i \le 0} - \delta_{j \le 0} = \begin{cases} 1, & \text{if } i \le 0 < j \\ -1, & \text{if } j \le 0 < i \\ 0, & \text{else} \end{cases}$$

Proposition 1: $\alpha(A,B) = \mathrm{Tr}(-A_{21}B_{12} + B_{21}A_{12}) \quad \forall A, B \in \bar{\mathfrak{a}}_\infty$, where we write any $X = \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right)$ w.r.t. $\{v_i\}_{i \le 0}$ & $\{v_i\}_{i > 0}$

Exercise: Prove this formula

Proposition 2: The above bilinear map $\alpha: \bar{\mathfrak{a}}_\infty \times \bar{\mathfrak{a}}_\infty \to \mathbb{C}$ is a 2-cocycle which is not a 2-coboundary. "Japanese 2-cocycle"

▸ • The cocycle condition is immediate from definition (use two Jacobi!)

• To show that α is nontrivial (i.e. not a 2-coboundary), it suffices to verify this for a restriction to any Lie subalgebra ↘

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(Continuation of proof).

Consider the abelian subalgebra $\bar{\mathcal{A}} \subseteq \bar{\mathfrak{a}}_\infty$ with basis $\{T^k\}_{k\in\mathbb{Z}}$, where
$T^k = \sum_{i\in\mathbb{Z}} E_{i,i+k}$ are translation operators

From Example 2) on page 3, we see that $\alpha(T^n, T^m) = n\delta_{n,-m}$. By Lectures 1-2 we know this is a nontrivial 2-cocycle that defines a Heisenberg algebra $\mathcal{A}$

Def 1: Let $\mathfrak{a}_\infty$ be the 1-dim central extension of $\bar{\mathfrak{a}}_\infty$ given by α:

$$\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}\cdot K$$

The above constructions can be summarized by the following result:

Theorem 1: Extend $\hat{\rho}$ to $\hat{\rho}: \mathfrak{a}_\infty \to \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V)$ via $\hat{\rho}(K) = \mathrm{Id}$. Then it gives rise to $\mathfrak{a}_\infty$-action on $\Lambda^{\frac{\infty}{2},m}V$

so it's not $\bar{\mathfrak{a}}_\infty$ but its central extension that acts on $\frac{\infty}{2}$-wedges!

Note: Extending $\mathbb{Z}$-grading on $\bar{\mathfrak{a}}_\infty$ to that on $\mathfrak{a}_\infty$ via $\deg(K)=0$, we see that $\Lambda^{\frac{\infty}{2},m}V$ (graded as on page 1) is a $\mathbb{Z}$-graded module$/\mathfrak{a}_\infty$.

Corollary 1: There is a Lie algebra embedding $\underset{\text{Heisenberg}}{\mathcal{A}} \hookrightarrow \mathfrak{a}_\infty$ via $a_n \mapsto T^n$, $K \mapsto K$

▸ this follows from above f-la $\alpha(T^n, T^m) = n\delta_{n,-m}$ ■

In particular, restricting $\mathfrak{a}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m}V$ to $\mathcal{A} \subseteq \mathfrak{a}_\infty$, we obtain

$$\boxed{\hat{\rho}: \mathcal{A} \curvearrowright \Lambda^{\frac{\infty}{2},m}V}$$

In fact, as we shall see now, it's not a new $\mathcal{A}$-module (however, this different realization is the subject of next 2 lectures!)

Proposition 3: For any $m\in\mathbb{Z}$, we have $\Lambda^{\frac{\infty}{2},m}V \simeq \underset{\text{Fock module}}{F_m}$ as $\mathcal{A}$-modules

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Proof of Prop 3

▶ Consider $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots \in \Lambda^{\frac{\infty}{2}, m} V$. Then: $T^n(\psi_m) = 0$ for $n>0$

Also $K(\psi_m) = \psi_m$ just b/c $\hat{\rho}(K) = \mathrm{Id}$. Finally, claim that $T^0(\psi_m) = m\psi_m$

- if $m>0$, then $T^0(\psi_m) = \hat{\rho}(\sum_{i\in\mathbb{Z}} E_{ii})(v_m \wedge v_{m-1} \wedge \ldots \wedge v_1 \wedge v_0 \wedge \ldots) = m \cdot v_m \wedge v_{m-1} \wedge \ldots$
- if $m \le 0$, then $T^0(\psi_m) = (\underbrace{(-1)+(-1)+ \ldots +(-1)}_{-m} + 0 + 0 + \ldots)\psi_m = m \cdot \psi_m$

Thus: we obtain an A-module morphism $F_m \xrightarrow{\sigma_m} \Lambda^{\frac{\infty}{2}, m} V$, $1 \longmapsto \psi_m$

As F_m is irreducible $\Rightarrow$ $\boxed{\sigma_m \text{ is injective}}$

Finally: As $\deg(1) = 0 = \deg(\psi_m)$, $\deg(a_n) = n = \deg(T^n)$, we see that σ_m preserves $\mathbb{Z}$-grading, but both spaces have the same characters (= generating series of dimensions): $\sum_{n\ge 0} \underbrace{p(n)}_{\#\text{ size } n \text{ Young diagrams}} q^n$, see page 1!

Hence: σ_m - isomorphism ∎

We have thus shown that $\Lambda^{\frac{\infty}{2}, m} V$ is already irreducible as A-module.

Define $\bar{\omega}_m \in (\mathfrak{a}_\infty^{(0)})^*$ via $\boxed{\bar{\omega}_m(K) = 1,\ \bar{\omega}_m(\sum_{i\in\mathbb{Z}} a_i E_{ii}) = \begin{cases} \sum_{1\le i\le m} a_i & , \text{if } m>0 \\ -\sum_{m+1\le i\le 0} a_i, & \text{if } m\le 0 \end{cases}}$

Proposition 4: $\Lambda^{\frac{\infty}{2}, m} V$ is irreducible highest weight representation of $\mathfrak{a}_\infty$ of highest weight $\bar{\omega}_m$, i.e. $\Lambda^{\frac{\infty}{2}, m} V \simeq L^+_{\bar{\omega}_m}$. Moreover, it's unitary

▶ Follows from above. Here $^\dagger : \mathfrak{a}_\infty \circlearrowleft$ is s.t. $K^\dagger = K$, $A^\dagger = A^t$ for $A \in \bar{\mathfrak{a}}_\infty$ ∎

Similarly to Corollary 1 of Lecture 9, we then obtain:

Corollary 2: For any $i_1, i_2, \ldots, i_N \in \mathbb{Z}$, the $\mathfrak{a}_\infty$-module $L^+_{\bar{\omega}_{i_1} + \ldots + \bar{\omega}_{i_N}}$ is unitary

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Besides for Heisenberg algebra, $\overline{a}_\infty$ also contains Virasoro algebras!

Recall: Problem 9 of Homework 1 provided a $\underset{\text{with algebra}}{\underbrace{W}}$-module $V_{\gamma,\beta}$ $\forall \beta,\gamma\in\mathbb{C}$.

Here $V_{\gamma,\beta} = \{g(t)t^\gamma (dt)^\beta \mid g(t)\in\mathbb{C}[t,t^{-1}]\}$, which has a basis $\{v_k\}_{k\in\mathbb{Z}}$ with $v_k = t^{-k+\gamma}\cdot(dt)^\beta$ (note: $k\rightsquigarrow -k$ vs part b) of that Problem)

Explicitly: the action is given by $\boxed{L_n(v_k) = (k-\gamma-(n+1)\beta)\, v_{k-n}}$

This provides a Lie algebra embedding

$$\boxed{\overline{\varphi}_{\gamma,\beta}: W \hookrightarrow \overline{a}_\infty \quad \text{given by} \quad L_n \mapsto \sum_{k\in\mathbb{Z}} (k-\gamma-(n+1)\beta)\, E_{k-n,k}}$$

$\boxed{\text{Exercise}}$: $\alpha(\overline{\varphi}_{\gamma,\beta}(L_n), \overline{\varphi}_{\gamma,\beta}(L_m)) = \delta_{n,-m}\Big(\underbrace{\frac{n^3-n}{12}c_\beta}_{\text{multiple of Virasoro 2-cocycle}} + \underbrace{2n\cdot h_{\gamma,\beta}}_{\text{2-coboundary}}\Big)$, $c_\beta := -12\beta^2+12\beta-2$, $h_{\gamma,\beta} := \gamma(\gamma+2\beta-1)/2$

This computation implies the following:

$$\boxed{\begin{array}{l}\text{Proposition 5: For any } \gamma,\beta\in\mathbb{C}\text{, there is a Lie algebra embedding}\\ \qquad \varphi_{\gamma,\beta}: Vir \hookrightarrow a_\infty \ \text{ s.t. } \ L_n \mapsto \overline{\varphi}_{\gamma,\beta}(L_n) + \delta_{n,0}\cdot h_{\gamma,\beta}\cdot K,\ C \mapsto c_\beta\cdot K\end{array}}$$

In particular, restricting $\hat{\rho}$ to this Virasoro, get $Vir \curvearrowright \Lambda^{\frac{\infty}{2},m}V$, charge "$c_\beta$" going back to [Feigin-Fuchs, 1982]. However, as † on a_∞ and Vir are not compatible we cannot say anything about unitarity!

$\boxed{\text{Exercise}}$: Verify that $L_k(\psi_m)=0$ for $k>0$, $L_0(\psi_m) = \frac{(\gamma-m)(\gamma+2\beta-1-m)}{2}\psi_m$

This computation implies:

$$\boxed{\begin{array}{l}\text{Corollary 3: For } \gamma,\beta\in\mathbb{C}\text{, set } \lambda := \left(\frac{(\gamma-m)(\gamma+2\beta-1-m)}{2},\ -12\beta^2+12\beta-2\right).\\ \qquad \text{Then there is a } Vir\text{-module morphism } \begin{array}{ccc} M_\lambda^+ & \longrightarrow & \Lambda^{\frac{\infty}{2},m}V \\ \Psi & & \Psi \\ v_\lambda^+ & \longmapsto & \psi_m \end{array}\end{array}}$$