Lecture #11

Last time: $\bar{\mathfrak{a}}_\infty \xrightarrow[\text{2-cocycle}]{} \underbrace{\mathfrak{a}_\infty}_{\mathcal{A} \quad Vir} \curvearrowright \Lambda^{\frac{\infty}{2},m}V$ and resulting $\mathcal{A} \curvearrowright \Lambda^{\frac{\infty}{2},m}V$ is $\simeq F_m$.

[**Remark:** Restriction of the 2-cocycle α to $\mathfrak{gl}_\infty$ is trivial, i.e. is a 2-coboundary.
Indeed, if $A, B \in \mathfrak{gl}_\infty$ and we use block decomposition from last time $A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22}\end{array}\right)$, then $\alpha(A,B) = \mathrm{Tr}\left(\left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & 0\end{array}\right)\cdot [A,B]\right)$

$\Rightarrow \alpha$ is a 2-coboundary!

(for students not asleep: what breaks down in $\bar{\mathfrak{a}}_\infty$ setup?)]

This week: We are going to investigate the aforementioned isomorphism $\Lambda^{\frac{\infty}{2},m} \simeq F_m$

Notations: $\mathcal{B}^{(m)} = F_m = \mathbb{C}[x_1, x_2, \ldots] \rightsquigarrow \mathcal{B} := \bigoplus_{m\in\mathbb{Z}} \mathcal{B}^{(m)} = \mathbb{C}[x_1,x_2,\ldots]\otimes\mathbb{C}[z,z^{-1}]$

$\mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2},m}V \rightsquigarrow \mathcal{F} := \bigoplus_{m\in\mathbb{Z}} \Lambda^{\frac{\infty}{2}}V$

think of $\mathcal{B}^{(m)}$ as $z^m\,\mathbb{C}[x_1,x_2,\ldots]$

Then, we can summarize the above as having

$\boxed{\mathcal{F} \xrightarrow[\sigma]{\sim} \mathcal{B}}$ - isomorphism of $\mathcal{A}$-modules

Terminology: $\mathcal{B}$ is called "bosonic space", $\mathcal{F}$ is called "fermionic space" } above isomorphism is known as "Boson - Fermion" correspondence

Today: How to extend $\mathcal{A} \curvearrowright \mathcal{B}$ to the action $\mathfrak{a}_\infty \curvearrowright \mathcal{B}$?
(which is obtained from $\mathfrak{a}_\infty \curvearrowright \mathcal{F}$ via σ)

Thursday: Describe $\sigma(v_{i_0}\wedge v_{i_1}\wedge\ldots)$, i.e. images of elementary $\frac{\infty}{2}$-wedges

Caution: The formulas we are going to derive today may look slightly different from [Kac-Raina, §5], but this is only b/c we used $a_{-j} \mapsto x_j$, $a_j \mapsto j\partial_{x_j}$ $(j>0)$ and [Kac-Raina] have $\begin{cases} a_{-j} \mapsto j x_j \\ a_j \mapsto \partial_{x_j}\end{cases}$

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Def 1: a) For $i\in\mathbb{Z}$, define the *wedging operator* $\xi_i=\hat{v}_i:\mathcal{F}\to\mathcal{F}$ via

$$v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots \longmapsto v_i\wedge v_{i_0}\wedge v_{i_1}\wedge\ldots$$

(here we get 0 in RHS if $i\in\{i_0,i_1,\ldots\}$, and otherwise $\pm$(elementary $\infty/2$-wedge))

b) For $i\in\mathbb{Z}$, define the *contraction operator* $\xi_i^*=\check{v}_i:\mathcal{F}\to\mathcal{F}$ via

$$v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots \longmapsto \begin{cases} 0 & \text{if } i\notin\{i_0,i_1,\ldots\} \\ (-1)^k v_{i_0}\wedge\ldots\wedge v_{i_{k-1}}\wedge v_{i_{k+1}}\wedge\ldots & \text{if } i=i_k\end{cases}$$

Note: For any $i,m\in\mathbb{Z}$, we have $\hat{v}_i:\mathcal{F}^{(m)}\to\mathcal{F}^{(m+1)}$, $\check{v}_i:\mathcal{F}^{(m)}\to\mathcal{F}^{(m-1)}$ (that's why we really want to work with the whole $\mathcal{F}$ today)

Exercise Verify $\hat{v}_i\hat{v}_j+\hat{v}_j\hat{v}_i=0$, $\check{v}_i\check{v}_j+\check{v}_j\check{v}_i=0$, $\check{v}_i\hat{v}_j+\hat{v}_j\check{v}_i=\delta_{ij}$ $\forall i,j\in\mathbb{Z}$

Now evoking previous 2 classes we see that

$$\boxed{\rho(E_{ij})=\xi_i\xi_j^* \implies \hat{\rho}(E_{ij})=\begin{cases}\xi_i\xi_j^*-1 & \text{if } i=j\leq 0\\ \xi_i\xi_j^* & \text{else}\end{cases}}$$

this can be taken as definition of $:\xi_i\xi_j^*:$

Hence, to express action of E_{ij} on $\mathcal{B}$ (which corresponds to $\hat{\rho}(E_{ij})$ via σ) it suffices to express both ξ_i, ξ_j^* via $\{\hat{\rho}(a_k)\mid k\in\mathbb{Z}\}$ — Heisenberg generators.

To this end, we shall work with quantum fields:

Def 2: $X(u):=\sum_{n\in\mathbb{Z}}\xi_n u^n\in\mathrm{End}(\mathcal{F})[[u,u^{-1}]]$ $\quad\xrightarrow{\sigma\text{-isom}}\quad$ $\Gamma(u)=\sigma X(u)\sigma^{-1}\in\mathrm{End}(\mathcal{B})[[u,u^{-1}]]$

$X^*(u):=\sum_{n\in\mathbb{Z}}\xi_n^* u^{-n}\in\mathrm{End}(\mathcal{F})[[u,u^{-1}]]$ $\qquad$ $\Gamma^*(u)=\sigma X^*(u)\sigma^{-1}\in\mathrm{End}(\mathcal{B})[[u,u^{-1}]]$

The main result for today is:

Theorem 1: For any $m\in\mathbb{Z}$, restrictions of $\Gamma(u)$, $\Gamma^*(u)$ to $\mathcal{B}^{(m)}$ are given by:

$$\Gamma(u)=u^{m+1}\cdot z\cdot\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(-\sum_{j>0}\frac{a_j}{j}u^{-j}\right)$$

$$\Gamma^*(u)=u^{-m}\cdot z^{-1}\cdot\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right)$$

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Before proceeding to the proof, let's clarify the right-hand sides above:

- the factor $z^{\pm 1}$ just to indicate the result will have coeff-s in $B^{(m\pm 1)}$
- exponents above are understood as formal series $\exp(X) = \sum_{n\geq 0} \frac{1}{n!} X^n$ and there is no ambiguity as $\{a_j\}_{j>0}$ (resp. $\{a_j\}_{j<0}$) pairwise commute
- while the product of two such exponents as in Right-hand sides is not well-defined as a series in u with coeff-s in $\mathcal{U}(A)$, but it's well-defined with coefficients in $\mathrm{End}(B)$ – explain why.

Remark: Evoking that $a_0|_{B^{(m)}} = m \cdot \mathrm{Id}$, the formulas can be written as:

$$\Gamma(u) = uz \cdot :\exp \int a(u)du: \quad \text{and} \quad \Gamma^*(u) = z^{-1} :\exp\left(-\int a(u)du\right): ,$$

where $a(u) = \sum_{j\in\mathbb{Z}} a_j u^{-j-1}$ and we use the same normal ordering $: \ :$ as before, e.g. $:\exp(\int a(u)du): = \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot u^m \cdot \exp\left(\sum_{j>0} \frac{a_j}{-j} u^{-j}\right)$

We start with a simple lemma:

Lemma 1: a) $[a_j, \Gamma(u)] = u^j \Gamma(u) \quad \forall j \in \mathbb{Z}$

b) $[a_j, \Gamma^*(u)] = -u^j \Gamma^*(u) \quad \forall j \in \mathbb{Z}$

► Pulling this back to the fermionic side, the equality a) is equivalent to:

$$[\hat{\rho}(T^j), X(u)] = u^j X(u)$$

But: $[\hat{\rho}(T^j), X(u)] = [\sum_{i\in\mathbb{Z}} \hat{\rho}(E_{i,i+j}), X(u)] = [\sum_{i\in\mathbb{Z}} \rho(E_{i,i+j}), X(u)] = [\sum_{i\in\mathbb{Z}} \xi_i \xi^*_{i+j}, \sum_n \xi_n u^n] =$

Note: $[\xi_i \xi^*_{i+j}, \xi_n] = \xi_i(-\xi_n \xi^*_{i+j} + \delta_{i+j,n}) - \xi_n \xi_i \xi^*_{i+j} = \delta_{i+j,n} \cdot \xi_i$ as $\xi_i \xi_n + \xi_n \xi_i = 0$

$= \sum_{i,n\in\mathbb{Z}} \delta_{i+j,n} \xi_i u^n = \sum_{n\in\mathbb{Z}} \xi_{n-j} u^n = u^j \cdot \sum_{n\in\mathbb{Z}} \xi_{n-j} u^{n-j} = u^j X(u)$ ✓

Exercise: Verify b) using the same logic

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Lemma 2: If α, β are two elements in an algebra s.t. $[\alpha,\beta]$ commutes with β, then $[\alpha, P(\beta)] = [\alpha,\beta]\cdot P'(\beta)$ for any polynomial $P(t)\in \mathbb{C}[t]$

▸ Suffices to check for $P(t)=t^n$ $\forall n$.

$$[\alpha,\beta^n] = [\alpha,\beta]\beta^{n-1} + \beta[\alpha,\beta]\beta^{n-2} + \ldots + \beta^{n-2}[\alpha,\beta]\beta + \beta^{n-1}[\alpha,\beta] \overset{\text{by assumption}}{=} [\alpha,\beta]\cdot n\beta^{n-1} = [\alpha,\beta](t^n)'|_{t=\beta}$$
■

We shall be using this result in the following special case:

Corollary 1: $[a_j, \exp(-\frac{a_j}{j}u^j)] = u^j\cdot \exp(-\frac{a_j}{j}u^j)$ as $\mathrm{End}(B)[[u^{-1}]]$

▸ As $[a_{-j}, a_j] = -j$ commutes with a_j, we get by above lemma ($\alpha=a_{-j}, \beta=a_j$):

$$[a_{-j}, \exp(-\frac{u^j}{j}\cdot a_j)] = (-j)\cdot(-\frac{u^j}{j})\exp(-\frac{u^j}{j}a_j) = u^j\exp(-\frac{u^j}{j}a_j)$$
(here P is series $P(t)=\exp(-\frac{u^j}{j}t)$)
■

Remark: Up to algebra automorphism $a_j \mapsto \frac{a_j}{j}$ $a_j \mapsto -j a_{-j}$ we can also think of this as equality of differential operators:

$$\left[\frac{\partial}{\partial x_j}, e^{u^j x_j}\right] = u^j\cdot e^{u^j x_j}$$ which is obvious!

Proof of Theorem 1

Step 1: Consider $\Gamma_+(u) := \exp(-\sum_{i>0}\frac{a_i}{i}u^{-i}) \in \mathrm{End}(B)[[u^{-1}]]$

Then: $[a_j, \Gamma_+(u)] = 0$ for $j>0$, but $[a_j, \Gamma_+(u)] = u^j\Gamma_+(u)$ by Corollary 1 (note that $[a_j,a_i]=0$ for $i\neq -j$)

But then, multiplying by $\Gamma_+(u)^{-1}$ on left & right, we get:

$$\boxed{[a_j, \Gamma_+(u)^{-1}] = 0 \text{ for } j>0, \quad [a_j, \Gamma_+(u)^{-1}] = -u^j\Gamma_+(u)^{-1} \text{ for } j<0}$$

Combining this with Lemma 1, we get for $\Delta(u) := z^{-1}\Gamma(u)\Gamma_+(u)^{-1}$:

$$\boxed{[a_j, \Delta(u)] = \begin{cases} u^j\Delta(u), & j>0 \\ 0, & j<0\end{cases}}$$

Moreover, f-la for $\Gamma(u)$ from Thm is equivalent to: $\boxed{\Delta(u) \overset{?}{=} u^{m+1}\exp(\sum_{i>0}\frac{a_{-i}}{i}u^i)}$

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(Continuation of the proof)

Step 2: Recovering $\Delta(u)$ up to prefactor.

Recall that $\Delta(u) \in \mathrm{End}(B^{(m)})[u,u^{-1}]$ commutes with $\{a_j \mid j<0\}$. However, entire $B^{(m)}$ is generated from $\mathbb{1}_m$ (= polynomial equal to 1) under $\{a_j \mid j<0\}$.

Hence, it suffices to determine $\Delta(u)\mathbb{1}_m$

But: For $j>0$, we have $a_j(\mathbb{1}_m)=0$ and $[a_j, \Delta(u)] = u^j \Delta(u)$ and a_j acts on $B^{(m)}$ via $a_j \mapsto j\partial_{x_j}$ $\Big\} \Rightarrow$

$$\Rightarrow \partial_{x_j}(\Delta(u)\mathbb{1}_m) = \frac{u^j}{j}\cdot \Delta(u)\mathbb{1}_m \quad \forall j>0 \Rightarrow \Delta(u)\mathbb{1}_m = F(u)\cdot \exp\Big(\sum_{j>0}\frac{u^j}{j}x_j\Big)$$

$$\Rightarrow \boxed{\Delta(u)\mathbb{1}_m = F(u)\exp\Big(\sum_{j>0}\frac{u^j a_{-j}}{j}\Big)\mathbb{1}_m} \text{ for some series } F(u)\in\mathbb{C}[[u,u^{-1}]]$$

$$\Downarrow$$

$$\boxed{\Delta(u) = F(u)\exp\Big(\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)}$$

Step 3: Determine the prefactor $F(u)$.

As we have equality, up to prefactor, we can find it by comparing any matrix coefficients! We shall pick the simplest vectors $\mathbb{1}_m \in B^{(m)}$, $\mathbb{1}_{m+1}\in B^{(m+1)}$

Then: $$\underbrace{\langle \mathbb{1}^*_{m+1} | \Gamma(u) | \mathbb{1}_m\rangle}_{\text{coeff. of } \mathbb{1}_{m+1} \text{ in } \Gamma(u)\mathbb{1}_m} = \langle \mathbb{1}^*_{m+1} | F(u)\cdot z\cdot \exp\Big(\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)\exp\Big(-\sum_{j>0}\frac{a_j}{j}u^{-j}\Big)|\mathbb{1}_m\rangle = \underline{\underline{F(u)}}$$

But on the other hand, pulling it back to fermionic side, we get:

$$\langle \mathbb{1}^*_{m+1} | \Gamma(u) | \mathbb{1}_m\rangle = \langle \psi^*_{m+1} | X(u) | \psi_m\rangle = \langle \psi^*_{m+1} | \sum v_n \wedge v_m \wedge v_{m-1}\wedge \ldots \cdot u^n\rangle = \underline{\underline{u^{m+1}}}$$

This proves $F(u) = u^{m+1}$, thus completing the proof of f-la for $\Gamma(u)$

$\boxed{\text{Exercise}}$: Prove the f-la for $\Gamma^*(u)$ using the same strategy

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To derive the formula for $\rho(E_{ij})$, $\hat{\rho}(E_{ij})$ via Heisenberg's action next time, we will need the following simple lemma & corollary:

Lemma 3: If α, β are two elements in an algebra such that $[\alpha,\beta]$ commutes with both α, β, then
$$e^{\alpha}\cdot e^{\beta} = e^{\beta}\cdot e^{\alpha}\, e^{[\alpha,\beta]}$$
as long as all are defined.

► By Baker-Campbell-Hausdorff f-la:
$$\left.\begin{aligned} e^{\alpha}\cdot e^{\beta} &= e^{\alpha+\beta+\frac{1}{2}[\alpha,\beta]+0\text{'s}} \\ e^{\beta}\cdot e^{\alpha} &= e^{\beta+\alpha+\frac{1}{2}[\beta,\alpha]+0\text{'s}} \end{aligned}\right\} \Rightarrow \text{claim!}$$

The context in which we shall apply this result is:

Corollary 2: $e^{a_j\cdot X}\cdot e^{a_{-j}Y} = e^{a_{-j}Y}\cdot e^{a_j X}\cdot e^{jXY}$ as operators on B for any $j>0$, X, Y - some constants

Exercise: Provide a direct proof of this f-la without using Lemma 3.