Lecture #12

Last time: $\Gamma(u)=u^{m+1}\cdot z\cdot \exp\Big(\sum\limits_{j>0}\frac{a_{-j}}{j}u^{j}\Big)\exp\Big(-\sum\limits_{j>0}\frac{a_j}{j}u^{-j}\Big) \;:\; B^{(m)}\longrightarrow B^{(m+1)}$

$\Gamma^*(u)=u^{-m}\cdot z^{-1}\cdot \exp\Big(-\sum\limits_{j>0}\frac{a_{-j}}{j}u^{j}\Big)\exp\Big(\sum\limits_{j>0}\frac{a_j}{j}u^{-j}\Big) \;:\; B^{(m)}\longrightarrow B^{(m-1)}$

↑ this is the first part of "boson – fermion correspondence".

Before we continue today to the 2^{nd} part, let's present the formula for the action (both ρ or $\hat{\rho}$) of E_{ij} (viewed as elt of $\mathfrak{gl}_\infty$ or $\bar{\mathfrak{a}}_\infty$).

Proposition 1: Let $\Gamma(u,v)=\exp\Big(\sum\limits_{j>0}\frac{u^j-v^j}{j}a_{-j}\Big)\exp\Big(-\sum\limits_{j>0}\frac{u^{-j}-v^{-j}}{j}a_j\Big)$

(a) $\rho\Big(\sum\limits_{i,j\in\mathbb{Z}}u^iv^{-j}E_{ij}\Big)=\dfrac{(u/v)^m}{1-v/u}\cdot\Gamma(u,v)$

(b) $\hat{\rho}\Big(\sum\limits_{i,j\in\mathbb{Z}}u^iv^{-j}E_{ij}\Big)=\dfrac{1}{1-v/u}\Big(\big(\tfrac{u}{v}\big)^m\Gamma(u,v)-1\Big)$

} on $B^{(m)}$ or $F^{(m)}$

► First, we note that (a) $\Rightarrow$ (b), since

$$\text{LHS of (b)}-\text{LHS of (a)}=-\sum_{n\geq 0}\frac{v^n}{u^n}=-\frac{1}{(1-v/u)} \quad\leftarrow \text{as geometric progression}$$

To prove (a), recall that $\rho(E_{ij})=\xi_i\xi_j^* \Rightarrow \rho\Big(\sum\limits_{i,j\in\mathbb{Z}}u^iv^{-j}E_{ij}\Big)=X(u)X^*(v)$

Switching to the bosonic side, we obtain:

$$\Gamma(u)\Gamma^*(v)=u^{(m-1)+1}z\exp\Big(\sum_{j>0}\frac{a_{-j}}{j}u^j\Big)\exp\Big(-\sum_{j>0}\frac{a_j}{j}u^{-j}\Big)v^{-m}z^{-1}\exp\Big(-\sum_{j>0}\frac{a_{-j}}{j}v^j\Big)\exp\Big(\sum\frac{a_j}{j}v^{-j}\Big)$$

$$=\Big(\frac{u}{v}\Big)^m\cdot G(u,v)\cdot\Gamma(u,v),$$

where $G(u,v)\underset{\uparrow}{=}\exp\Big(\sum\limits_{j>0}\frac{j}{j^2}\cdot\Big(\frac{v}{u}\Big)^j\Big)=\exp\Big(-\ln\Big(1-\frac{v}{u}\Big)\Big)=\dfrac{1}{1-v/u}$

by Lemma 3 of Lecture 11 ∎

Today: Second part of "boson – fermion correspondence".

Q: Evaluate the images of $\frac{\infty}{2}$-wedges $v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\dots$ in the bosonic side.

Def 1: For x a collection of variables $(x_1, x_2, \ldots)$ define $S_k(x) \in \mathbb{C}[x_1, x_2, \ldots]$ via
$$\sum_{k\geq 0} S_k(x)\cdot z^k = \exp\Big(\sum_{i\geq 1} x_i z^i\Big)$$

They are closely related to complete symmetric functions $h_k(t)$:
$$h_k(t) = h_k(t_1, \ldots, t_N) = \sum_{\substack{p_1,\ldots,p_N\geq 0\\ p_1+\ldots+p_N=k}} t_1^{p_1} t_2^{p_2} \ldots t_N^{p_N}$$
as explained in:

Lemma 1: Fix N and consider $t_1, \ldots, t_N$. Substituting $x_n = \frac{t_1^n + \ldots + t_N^n}{n}$ $\forall n$, get
$$S_k(x) = h_k(t)$$

$$\Rightarrow \sum_{k\geq 0} S_k(x) z^k = \exp\Big(\sum_{n\geq 1} \frac{t_1^n + \ldots + t_N^n}{n} z^n\Big) = \exp\Big(-\sum_{j=1}^N \log(1-t_j z)\Big) = \prod_{j=1}^N \frac{1}{1-t_j z} = \sum_{k\geq 0} h_k(t) z^k$$

Def 2: To any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0)$, define (bosonic) Schur polynomial
$$S_\lambda(x) = \det\big(S_{\lambda_i+j-i}(x)\big) = \det\begin{pmatrix} S_{\lambda_1}(x) & S_{\lambda_1+1}(x) & \ldots & S_{\lambda_1+m-1}(x) \\ S_{\lambda_2-1}(x) & S_{\lambda_2}(x) & \ldots & S_{\lambda_2+m-2}(x) \\ \vdots & & & \\ S_{\lambda_m-m+1}(x) & \ldots & & S_{\lambda_m}(x) \end{pmatrix}$$
where $S_{<0}(x) = 0$

Remark: a) $S_\lambda(x)$ does not change after adding several 0's at the end

b) Specializing $x_n = \frac{t_1^n + \ldots + t_N^n}{n}$ $\forall n$, we get the classical Schur polynomial, which is symmetric in $t_1, \ldots, t_N$, and equals
$$S_\lambda(t) = \det\big(h_{\lambda_i+j-i}(t)\big) \quad \leftarrow \text{"Jacobi-Trudi" formula}$$

c) These symmetric Schur functions can also be realized as traces of $\mathrm{diag}(t_1, \ldots, t_N) \in GL_N$ acting on irreducible GL_N-module of highest weight λ (assuming $m \leq N$ and we added $N-m$ of 0's at the end of λ.

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Warning: To state the result below, we shall need to slightly rescale action $\mathcal{A} \curvearrowright F_0$ via $a_j \mapsto \partial_{x_j}$, $a_{-j} \mapsto j x_j$ $(j>0)$.

Otherwise, if using our earlier conventions $(a_j \mapsto j\partial_{x_j},\ a_{-j} \mapsto x_j)$ we would end up with a more complicated answer.

Recall: Basis of $\mathcal{F}^{(0)}$ is $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots$ parametrized by $i_0 > i_1 > i_2 > \ldots$ s.t. $i_k = -k$ for $k \gg 0$, which are in bijection with partitions

$$\boxed{\lambda = (i_0, i_1 + 1, i_2 + 2, \ldots)}$$

↖ only finitely many $\neq 0$.

Theorem 1: $\sigma(\underbrace{v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots}_{\in \mathcal{F}^{(0)}}) = \underset{\in \mathcal{B}^{(0)}}{S_\lambda(x)}$ for $i_0 > i_1 > i_2 > \ldots$ and λ as above

▶ Let $P(x) := \sigma(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots)$. Consider another set of variables $\{y_i | i \geq 1\}$.

Evaluating $\langle \mathbb{1}, e^{y_1 a_1 + y_2 a_2 + \ldots} P(x) \rangle = \langle \mathbb{1}, e^{y_1 \partial_{x_1} + y_2 \partial_{x_2} + \ldots} P(x) \rangle \underset{\text{Taylor series}}{=} \langle \mathbb{1}, P(x+y) \rangle = P(y)$

But on fermionic side, this equals:

$$\langle \psi_0, e^{y_1 T + y_2 T^2 + \ldots} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) \rangle = \langle \psi_0, \underbrace{\Big(\sum_{k \geq 0} S_k(y) T^k\Big)(v_{i_0} \wedge v_{i_1} \wedge \ldots)}\rangle$$

which is precisely the coefficient of the $\frac{\infty}{2}$-wedge ψ_0 in

The latter is precisely the bosonic Schur polynomial $S_\lambda(y)$.

(discuss in class, starting with finite dim. analogue)

So: $P(y) = S_\lambda(y)$ as claimed ▩

Remark: Likewise, using the same argument, one proves that

$$\sigma(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) = S_\lambda(x)$$

for any $i_0 > i_1 > i_2 > \ldots$ s.t. $i_k = m - k$ for $k \gg 0$

and $\lambda = (i_0 - m, i_1 + 1 - m, i_2 + 2 - m, \ldots)$

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We shall end today's class with the notion of semi-infinite Grassmannian that we shall need next time. To motivate, we start with fin.dim. case.

Setup: V - fin. dim. vector space $/\mathbb{C}$, $\{v_1, \ldots, v_n\}$ - basis of V, $0 \le k \le n$

Def 3: Let $\Omega = GL(V)(v_1 \wedge v_2 \wedge \ldots \wedge v_k) = \{g(v_1) \wedge g(v_2) \wedge \ldots \wedge g(v_k) \mid g \in GL(V)\}$

$\boxed{\text{Exercise}}$: Verify that $\Omega = \{\text{nonzero decomposable wedges}\}$
$= \{x_1 \wedge x_2 \wedge \ldots \wedge x_k \mid x_1, \ldots, x_k \in V - \text{lin. indep.}\}$

Def 4: The k-Grassmannian of V, denoted $Gr(k,V) = Gr(k,n)$, is the set of all k-dimensional subspaces of V.

In fact, $Gr(k,V)$ is a projective variety, which is based on

$$\boxed{\text{Plücker embedding } \; Pl: Gr(k,V) \longrightarrow \mathbb{P}(\Lambda^k V)}$$

$$\begin{pmatrix} k\text{-dim subspace} \\ \text{with a basis } x_1, \ldots, x_k \end{pmatrix} \longmapsto \text{projection of } x_1 \wedge \ldots \wedge x_k \text{ in the projectivization}$$

Thus: $Gr(k,V) \simeq \Omega/\mathbb{C}^\times$

The basic question one may ask is if given $\tau \in \Lambda^k V$ we can check $\tau \in \Omega$? To this end, we shall consider finite analogues of operators from Lecture 11:

- wedging operator $\hat{v}_i : \Lambda^k V \longrightarrow \Lambda^{k+1} V$
- contraction operator $\check{v}_i^* : \Lambda^k V \longrightarrow \Lambda^{k-1} V$, where $\{v_i^*\}$ - dual basis of V^*

and combine them to define

$$\boxed{S := \sum_{i=1}^n \hat{v}_i \otimes \check{v}_i^* \; : \; \Lambda^k V \otimes \Lambda^k V \longrightarrow \Lambda^{k+1} V \otimes \Lambda^{k-1} V}$$

$\boxed{\text{Exercise}}$: Verify that S is independent of basis $\{v_1, \ldots, v_n\}$

$\boxed{\text{Theorem 2 ("Plucker relations"): For } \tau \in \Lambda^k V \setminus \{0\}: \quad \tau \in \Omega \iff S(\tau \otimes \tau) = 0}$

We shall recall the proof of this next time.

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Let's now return back to ∞-dim setup, where V has a basis $\{v_i\}_{i\in\mathbb{Z}}$.

Def 5: a) Let $M(\infty) = \mathrm{Id} + \mathfrak{gl}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \mid a_{ij}-\delta_{ij}$ are zero but finitely many$\}$

b) Let $GL(\infty)$ be a subset of invertible elements in $M(\infty)$.

$\boxed{\text{Exercise}}$: a) Verify that $M(\infty)$ - monoid, $GL(\infty)$ - group.

b) Verify that $M(\infty) \curvearrowright \mathcal{F}^{(m)}$ via $A(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots) = Av_{i_0}\wedge Av_{i_1}\wedge Av_{i_2}\wedge\ldots$ and the group $GL(\infty)$ thus acts on $\mathcal{F}^{(m)}$.

c) Verify that for any $X\in\mathfrak{gl}_\infty$, $\exp(X)$ is in GL_∞.

Def 6: Define $\Omega := GL(\infty)(\psi_0) \subseteq \mathcal{F}^{(0)}$.

$\boxed{\text{Lemma 2: For any } i_0>i_1>i_2>\ldots \text{ with } i_k=-k, \text{ have } v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots\in\Omega}$

▷ $v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots = A(\psi_0)$ with $A = \sum_{k\in\mathbb{Z}} E_{\sigma(k),k}$, where $\sigma:\mathbb{Z}\to\mathbb{Z}$ is a bijective map s.t. $\sigma(k)=k$ if $|k|\gg 0$ and $\sigma(-j)=i_j\ \forall j\geq 0$ ∎

Finally, we define:

Def 7: The semi-infinite Grassmannian is $Gr := \Omega/\mathbb{C}^\times$

We conclude with an analogue of Theorem 2. To this end, we define

$$\boxed{S = \sum_{i\in\mathbb{Z}} \hat{v}_i \otimes \check{v}_i : \mathcal{F}^{(m)}\otimes\mathcal{F}^{(m)} \longrightarrow \mathcal{F}^{(m+1)}\otimes\mathcal{F}^{(m-1)}}$$

Note: For any $w_1\otimes w_2\in\mathcal{F}^{(m)}\otimes\mathcal{F}^{(m)}$, have $\left.\begin{array}{l}\hat{v}_i(w_1)=0 \text{ for } i\ll 0\\ \check{v}_i(w_2)=0 \text{ for } i\gg 0\end{array}\right\}\Rightarrow$

$\Rightarrow S(w_1\otimes w_2)$ is well-defined and is given by a finite sum.

$\boxed{\text{Theorem 3: For } \tau\in\mathcal{F}^{(0)}\setminus\{0\}, \text{ have: } \quad \tau\in\Omega \Leftrightarrow S(\tau\otimes\tau)=0}$

$\boxed{\text{Exercise}}$: Prove this result, using Thm 2.