Lecture #13

Last time: • 2^{nd} part of Boson-Fermion correspondence
• started discussing semi-infinite Grassmannian Gr

This week: Relate Gr to solutions of important PDE!

• But first, let's start by proving Theorem 2 from last time:

> **Theorem 1**: For n-dim V and $1\le k\le n$, $\tau\in\Lambda^k V\setminus\{0\}$:
> $$\tau\in\Omega \iff S(\tau\otimes\tau)=0$$

Here: $S=\sum_{i=1}^n \hat{v}_i\otimes\check{v}_i^*$ where $\{v_1,\dots,v_n\}$ - any basis of V, $\{v_1^*,\dots,v_n^*\}$ - dual basis.

($\Rightarrow$)
If $\tau=w_1\wedge\dots\wedge w_k$, then pick a basis $\{w_1,\dots,w_k,w_{k+1},\dots,w_n\}$ of V. Then $\hat{w}_i(\tau)=0$ for $i\le k$, while $\check{w}_j^*(\tau)=0$ for $j>k$. Hence: $S(\tau\otimes\tau)=0$.

($\Leftarrow$)
Assume $S(\tau\otimes\tau)=0$. Define subspaces $E\subseteq V$ and $F\subseteq V^*$ via

$$\boxed{E=\{v\in V\mid \hat{v}(\tau)=0\},\qquad F=\{f\in V^*\mid \check{f}(\tau)=0\}}$$

First, we note that $E\subseteq F^\perp:=\{v\in V\mid f(v)=0\ \forall f\in F\}$, due to $\hat{v}\check{f}+\check{f}\hat{v}=f(v)\mathrm{Id}$.
Let $r=\dim E$, $s=\dim F^\perp$. Pick a basis $\{v_1,\dots,v_n\}$ of V so that $\{v_1,\dots,v_r\}$ - basis of E, $\{v_1,\dots,v_r,\dots,v_s\}$ - basis of $F^\perp$. Then $\{v_{s+1}^*,\dots,v_n^*\}$ - basis of F.
Note that $\hat{v}_i(\tau)=0$ for $1\le i\le r$, $\check{v}_j(\tau)=0$ for $s<j\le n$. Hence:

$$\boxed{0=S(\tau\otimes\tau)=\sum_{i=r+1}^{s}\hat{v}_i(\tau)\otimes\check{v}_i^*(\tau)}$$

We claim that $r=s$. If not, i.e. $r<s$, then $\{v_i\wedge\tau\mid r<i\le s\}$ are all lin indep (as otherwise $(c_{r+1}v_{r+1}+\dots+c_sv_s)\wedge\tau=0$ for not all $c_j'=0 \Rightarrow c_{r+1}v_{r+1}+\dots+c_sv_s\in E \Rightarrow$ ↯)
But then we must have $\check{v}_i^*(\tau)=0\ \forall r<i\le s \Rightarrow v_i^*\in F\Rightarrow$ ↯.

So: $r=s$ and $E=F^\perp$. Write $\tau=\sum_{i_1<\dots<i_k} c_{i_1\dots i_k}\cdot v_{i_1}\wedge\dots\wedge v_{i_k}$

• $\hat{v}_i(\tau)=0\ \forall 1\le i\le r \Rightarrow \{i_1,\dots,i_k\}\ni 1,2,\dots,r$ if $c_{i_1\dots i_k}\neq 0$
• $\check{v}_j^*(\tau)=0\ \forall r<j\le n \Rightarrow \{i_1,\dots,i_k\}\not\ni r+1,\dots,n$ if $c_{i_1\dots i_k}\neq0$

$\Rightarrow k=r$ and $\tau=v_1\wedge\dots\wedge v_k\cdot\mathrm{const}$

Lecture #13

- Let's now see what $S(\tau\otimes\tau)=0$ looks like down-to-earth.

To this end, we fix a basis $\{v_1, \ldots, v_n\}$ of V, so that

$\{v_{i_1}\wedge v_{i_2}\wedge\ldots\wedge v_{i_k} \mid 1\le i_1<i_2<\ldots<i_k\le n\}$ – basis of $\Lambda^k V$.

Given any subspace W of V, $\dim(W)=k$, pick a basis $\{x_1, \ldots, x_k\}$ of W.
Each x_j $(1\le j\le k)$ is a linear combination of $\{v_i\}_{i=1}^n$, giving rise to $n\times k$ matrix $A=(a_{ij})_{1\le i\le n}^{1\le j\le k}$ so that $x_j=\sum_{i=1}^n a_{ij}v_i$. Then, $x_1\wedge\ldots\wedge x_k$ is written in above basis as:

$$\boxed{x_1\wedge x_2\wedge\ldots\wedge x_k=\sum_{\substack{I\subseteq\{1,\ldots,n\},\ |I|=k\\ \{i_1<\ldots<i_k\}}} p_I\cdot v_{i_1}\wedge\ldots\wedge v_{i_k}\quad\text{with}\quad p_I=\det(a_{ij})_{i\in I}^{1\le j\le k}}$$

Example: $n=4, k=2$, $A=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\\ a_{41} & a_{42}\end{pmatrix}$. Then

$$x_1\wedge x_2=(a_{11}v_1+a_{21}v_2+a_{31}v_3+a_{41}v_4)\wedge(a_{12}v_1+a_{22}v_2+a_{32}v_3+a_{42}v_4)=$$
$$=(a_{11}a_{22}-a_{21}a_{12})\cdot v_1\wedge v_2+(a_{11}a_{32}-a_{31}a_{12})v_1\wedge v_3+(a_{11}a_{42}-a_{41}a_{12})v_1\wedge v_4$$
$$+(a_{21}a_{32}-a_{31}a_{22})v_2\wedge v_3+(a_{21}a_{42}-a_{41}a_{22})v_2\wedge v_4+(a_{31}a_{42}-a_{41}a_{32})v_3\wedge v_4$$

The following is a coordinate form of Theorem 1 (known as "Plücker rel-s"):

Exercise: For $\tau=\sum_{1\le i_1<\ldots<i_k\le n} p_{i_1\ldots i_k}\cdot v_{i_1}\wedge\ldots\wedge v_{i_k}\in\Lambda^k V\setminus\{0\}$:

$$\tau\in\Omega \iff \sum_{J\ni j\notin I}(-1)^{\mu(j)+\nu(j)}p_{I\cup\{j\}}p_{J\setminus\{j\}}=0\qquad\forall\, I,J\subseteq\{1,\ldots,n\}\quad \begin{matrix}|I|=k-1\\ |J|=k+1\end{matrix}$$

Example: For $n=4$, $k=2$, there is essentially only 1 relation that reads
$$p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23}=0$$

E.g. if $(I=\{1\}, J=\{2,3,4\})$ or $(I=\{2\}, J=\{1,3,4\})$ or $(I=\{3\}, J=\{1,2,4\})$ or $\begin{pmatrix}I=\{4\}\\ J=\{1,2,3\}\end{pmatrix}$
then all of them give the same relation above. On the other hand, if $I\subseteq J$, that is $I=\{i\}$, $J=\{i,j,k\}$, then the above sum is just zero.

Down-to-earth, when expanding $S(\tau\otimes\tau)$ in the basis $\{v_{i_1}\wedge v_{i_2}\wedge v_{i_3}\otimes v_j\}_{i_1<i_2<i_3}$
all coeff-s will be either zero or $\pm(p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23})$

Lecture #13

Let's now move to the infinite-dimensional V with a basis $\{v_i\}_{i\in\mathbb{Z}}$

Recall from Lecture 12:

$$\boxed{\begin{aligned}&\Omega = GL(\infty)(v_0\wedge v_{-1}\wedge v_{-2}\wedge\ldots)\\ &Gr = \Omega/\mathbb{C}^\times\end{aligned}}$$

Rmk: Identifying V with $\mathbb{C}[t,t^{-1}]$ via $v_i \leftrightarrow t^{-i}$, it's easy to see that

$$Gr = \left\{\text{subspace } E\subseteq V \;\middle|\; \begin{matrix} t^N\mathbb{C}[[t]]\subseteq E \text{ for } N\gg 0\\ \dim(E/t^N\mathbb{C}[[t]])=N\end{matrix}\right\}$$

$$= \{\text{subspace } E\subseteq V \mid \exists N:\ t^N\mathbb{C}[[t]] \underset{\text{codim } N}{\subseteq} E \subseteq t^{-N}\mathbb{C}[[t]]\}$$

$$= \bigcup_{N\geq 1} Gr(N, 2N) \quad \text{"nested union"}$$

- For the rest of today, we shall rewrite the relation $S(\tau\otimes\tau)$ on the bosonic side. To this end, we recall that

$$\sum_{i\in\mathbb{Z}} \hat{v}_i\cdot u^i = X(u) = \sigma^{-1}(\Gamma(u)), \quad \sum_{i\in\mathbb{Z}} \check{v}_i\, u^{-i} = X^*(u) = \sigma^{-1}(\Gamma^*(u))$$

In particular: $S(\tau\otimes\tau)=0 \iff \boxed{CT_u\left(X(u)\tau\otimes X^*(u)\tau\right)=0}$

CT_u extracts coefficient of u^0.

On the bosonic side: $\tau\in B^{(0)}=\mathbb{C}[x_1,x_2,x_3,\ldots] \Rightarrow \tau\otimes\tau\in\mathbb{C}[x_1',x_1'',x_2',x_2'',\ldots]$ (where 1^{st} copy of $B^{(0)}$ is generated by $\{x_i'\}$, 2^{nd} copy – by $\{x_i''\}$)

Also: $X(u) \rightsquigarrow \Gamma(u) = u^{m+1}\cdot z\cdot \exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(\sum_{j>0}\frac{a_j}{-j}u^{-j}\right)$

$X^*(u)\rightsquigarrow \Gamma^*(u) = u^{-m}\cdot z^{-1}\cdot\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right)$

$a_{-j}\mapsto j x_j,\quad a_j\mapsto \partial_{x_j}\quad \forall j>0$

So: $\boxed{CT_u\left(u\cdot e^{\sum_{j>0}x_j'\cdot u^j}\cdot e^{-\sum_{j>0}\frac{1}{j}\partial_{x_j'}u^{-j}}\cdot e^{-\sum_{j>0}x_j''\cdot u^j}\cdot e^{-\sum_{j>0}\frac{1}{j}\partial_{x_j''}u^{-j}}\,\tau(x')\tau(x'')\right)=0}$

But $\{x_i'\}$ & $\{x_j''\}$ are independent sets of variables $\Rightarrow$ can reorder to get:

$$\boxed{CT_u\left(u\cdot\exp\Big(\sum_{j>0}(x_j'-x_j'')u^j\Big)\cdot\exp\Big(\sum_{j>0}(\partial_{x_j'}-\partial_{x_j''})\frac{u^{-j}}{-j}\Big)\tau(x')\tau(x'')\right)=0}$$

Lecture #13

The above last relation suggest the following change of variables:

$$\begin{cases} x' = x - y \\ x'' = x + y \end{cases}, \text{ i.e. } \begin{matrix} x_i' = x_i - y_i \\ x_i'' = x_i + y_i \end{matrix} \quad \forall i \geq 1$$

Then: $x_j' - x_j'' = -2y_j$ and $\partial_{x_j'} - \partial_{x_j''} = -\partial_{y_j}$ (by chain rule)

Plugging this, we obtain:

$$\boxed{CT_u\left(u \cdot \exp\left(\sum_{j>0} -2y_j \cdot u^j\right) \cdot \exp\left(\sum_{j>0} \frac{1}{j} \partial_{y_j} \cdot u^{-j}\right) \cdot \tau(x-y)\tau(x+y)\right) = 0}$$

Def 1: For any $f, g \in \mathbb{C}[x_1, x_2, x_3, \ldots]$ and $P \in \mathbb{C}[[x_1, x_2, x_3, \ldots]]$, set

$$A(P, f, g) := \left[P\left(\frac{\partial}{\partial z}\right)\left(f(x-z)g(x+z)\right)\right]\Big|_{z=0}$$

Exercise: Verify $A(P, f, f) = 0 \ \forall f$ if P is odd, i.e. $P(x) = -P(-x)$

Then, the above relation can be written as follows (Kashiwara–Miwa):

Theorem 2 (Hirota bilinear relations): For $\tau \in B^{(0)} \setminus \{0\}$, $\sigma^{-1}(\tau) \in \Omega$ iff

$$A\left(\sum_{j\geq 0} S_j(-2y) S_{j+1}(\tilde{x}) \exp\left(\sum_{s\geq 1} y_s x_s\right), \tau, \tau\right) = 0 \quad \text{where } \tilde{x}_i = \frac{x_i}{i}$$

Here, S_j's were introduced in Lecture 12, so that

$$\sum_{j\geq 0} S_j(-2y) u^j = \exp\left(-\sum_{j\geq 1} 2y_j u^j\right)$$

$$\sum_{j\geq 0} S_j\left(\frac{1}{j}\partial_{y_j}\right) u^{-j} = \exp\left(\sum_{j\geq 1} \frac{\partial_{y_j}}{j} u^{-j}\right)$$

Note: The above is actually an infinite family of relations, each obtained by evaluating the coefficient of a certain y-monomial in the above $A(\ldots, \tau, \tau)$.

We shall prove this result and discuss consequences next time