Lecture #14

Last time we ended with the following result:

Theorem 1 (Hirota bilinear relations): For $\tau \in B^{(0)} \setminus \{0\}$, we have $\sigma^{-1}(\tau) \in \Omega \subseteq \mathcal{F}^{(0)}$ iff

$$A\Big(\sum_{j\geq 0} S_j(-2y)\, S_{j+1}(\tilde{x}) \exp\big(\sum_{s\geq 0} y_s x_s\big), \tau, \tau\Big) = 0 \quad \text{, where } \tilde{x}_k = \tfrac{1}{k} x_k \ \forall k$$

► By what we derived last time: $\sigma^{-1}(\tau) \in \Omega$ iff:

$$\underbrace{CT_u\Big(u \cdot \exp\big(-\sum_{j\geq 1} 2y_j \cdot u^j\big) \exp\big(\sum_{j\geq 1} \tfrac{1}{j} \partial_{y_j} \cdot u^{-j}\big)\, \tau(x+y)\tau(x-y)\Big)}_{\shortparallel} = 0$$

$$CT_u\Big(u \cdot \sum_{j\geq 0} S_j(-2y) u^j \cdot \sum_{j\geq 0} S_j(\tilde{\partial}_y) u^{-j} \cdot \tau(x+y+t)\tau(x-y-t)\Big)\Big|_{t=0}$$

$\shortparallel$ since $f(\tilde{\partial}_y)\, g(y+t) = f(\partial_t)\, g(y+t) \ \forall f, g$

$$\Big(\sum_{j\geq 0} S_j(-2y) S_{j+1}(\tilde{\partial}_t)\, \tau(x+y+t)\tau(x-y-t)\Big)\Big|_{t=0}$$

$\shortparallel$ Taylor series expansion

$$\Big(\sum_{j\geq 0} S_j(-2y) S_{j+1}(\tilde{\partial}_t) \exp\big(\sum_{s\geq 1} y_s \partial_{t_s}\big)\, \tau(x+t)\tau(x-t)\Big)\Big|_{t=0}$$

$$\shortparallel$$

$$A\big(\textstyle\sum_{j\geq 0} S_j(-2y) S_{j+1}(\tilde{x}) \exp(\sum_{s\geq 1} y_s x_s), \tau, \tau\big)$$

As noted last time, this is actually an infinite family of relations, each obtained by reading off a certain monomial $= y^{\gamma}$. Also, recall

(*) $A(P, \tau, \tau) = 0 \ \ \forall$ odd P (exercise of Lect 13)

• $\gamma = 0$, i.e. $y^{\gamma} = 1$.

Then Hirota eq-n gives $A(x_1, \tau, \tau) = 0$, but this already follows from (*)

• $\gamma = (0, \ldots 0, \underset{r}{1}, 0 \ldots)$, i.e. $y^{\gamma} = y_r$

Then Hirota eq-n gives $\underbrace{A(x_1 x_r - 2 S_{r+1}(\tilde{x}), \tau, \tau) = 0 \quad \forall r \geq 1}_{\text{known as a "KP hierarchy"}}$

- For $r=1$: $x_1^2 - 2S_2(\tilde{x}) = -x_2 \Rightarrow$ follows by (*)
- For $r=2$: $x_1 x_2 - 2S_3(\tilde{x}) = -\frac{x_1^3}{3} - \frac{2}{3}x_3$ - odd $\Rightarrow$ follows by (*)
- For $r=3$: $x_1 x_3 - 2S_4(\tilde{x}) = \frac{1}{3}x_1 x_3 - \underset{\sim}{\frac{1}{2}x_4} - \frac{x_2^2}{4} - \frac{x_1^4}{12} - \underset{\sim}{\frac{x_1^2 x_2}{2}}$ (with $\sim$ - odd)

$$\Downarrow \quad \boxed{A\big(\tfrac{1}{3}x_1 x_3 - \tfrac{1}{4}x_2^2 - \tfrac{1}{12}x_1^4, \tau, \tau\big) = 0}$$

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The relation from the end of p. 1 gives:

$$\boxed{\left((\partial_{z_1}^4 + 3\partial_{z_2}^2 - 4\partial_{z_1}\partial_{z_3})\tau(x-z)\tau(x+z)\right)\Big|_{z=0} = 0} \qquad (\dagger)$$

To relate this to something studied in PDE, we change the variables $x_1 \mapsto x$, $x_2 \mapsto y$, $x_3 \mapsto t$, $x_m \mapsto c_m$ $(m>3)$ – can be viewed as constants.

Set: $\quad u = 2\partial_x^2 \log \tau$

Exercise: τ satisfies $(\dagger)$ iff u satisfies $\underbrace{\frac{3}{4}\partial_y^2 u = \partial_x\left(u_t - \frac{3}{2}\cdot u u_x - \frac{1}{4}u_{xxx}\right)}_{\text{KP equation}}$

Exercise Verify that $x_1 x_r - 2S_{r+1}(\tilde{x})$ is not odd for $r>2$ $\Rightarrow$ gives nontrivial eq-n

Corollary 1: Any element of $Gr = \Omega/\mathbb{C}^\times$ gives a solution of KP equation. In particular, $2\partial_x^2 \log S_\lambda(x,y,t,c_{\geq 4})$ is a sol-n for any partition λ

- We shall next construct more solutions of KP eq-n. To this end:

Lemma 1: S commutes with gl_∞-action

▷ Recall $E_{k\ell} = \hat{v}_k \check{v}_\ell$ and $\Delta(E_{k\ell}) = E_{k\ell}\otimes 1 + 1\otimes E_{k\ell}$, while $S = \sum_{i\in\mathbb{Z}} \hat{v}_i \otimes \check{v}_i$.

$[\hat{v}_k\check{v}_\ell \otimes 1, \sum_i \hat{v}_i\otimes\check{v}_i] = \sum_i (\hat{v}_k(-\hat{v}_i\check{v}_\ell + \delta_{i\ell}) - \hat{v}_i\hat{v}_k\check{v}_\ell)\otimes\check{v}_i = \hat{v}_k\otimes\check{v}_\ell$

$[1\otimes\hat{v}_k\check{v}_\ell, \sum\hat{v}_i\otimes\check{v}_i] = \sum_i \hat{v}_i\otimes(\hat{v}_k\check{v}_\ell\check{v}_i - (-\hat{v}_k\check{v}_i\check{v}_\ell + \delta_{ik}\check{v}_\ell)) = -\hat{v}_k\otimes\check{v}_\ell$

Hence: $[\Delta(E_{k\ell}), S] = 0$, and $\{E_{k\ell}\}$ – basis of gl_∞ ∎

Let's now recall the vertex operators from Lectures 11-12:

$$\Gamma(u)|_{B^{(-1)}} = z\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\exp\left(-\sum_{j>0}\frac{a_j}{j}u^{-j}\right)$$

$$\Gamma^*(v)|_{B^{(0)}} = z^{-1}\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}v^j\right)\exp\left(\sum_{j>0}\frac{a_j}{j}v^{-j}\right)$$

$$\Gamma(u,v)|_{B^{(0)}} =: :\Gamma(u)\Gamma^*(v):|_{B^{(0)}} = \exp\left(\sum_{j>0}\frac{u^j-v^j}{j}a_{-j}\right)\exp\left(-\sum_{j>0}\frac{u^{-j}-v^{-j}}{j}a_j\right)$$

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Our second main result today is:

$$\boxed{\text{Theorem 2: If } \tau \in \Omega, \text{ then } (1+a\Gamma(u,v))\tau \in \Omega_{u,v} := \{\tau' \in B^{(0)}((u,v)) \mid S(\tau \otimes \tau') = 0\} \quad \forall a \in \mathbb{C}}$$

In particular, iterating this several times, we get:

$$\boxed{\text{Corollary 2: } (1+a_1\Gamma(u_1,v_1))\ldots(1+a_n\Gamma(u_n,v_n))\mathbb{1} \in \Omega_{u_1,v_1,\ldots,u_n,v_n} \quad \forall a_1,\ldots,a_n \in \mathbb{C}}$$

see [Kac-Raina, Cor 7.2] and discussion after it that recovers the "n-soliton" solution of the KP eq-n.

Example: For $n=1$, get $\tau = 1 + a\cdot e^{(u-v)x + (u^2-v^2)y + (u^3-v^3)t + \ldots} = 1 + e^{(u-v)x+(u^2-v^2)y+(u^3-v^3)t+c}$

Then: $u(x,y,t) = 2\partial_x^2 \log\tau = \dfrac{(u-v)^2}{2} \cdot \dfrac{1}{\cosh^2 \frac{1}{2}((u-v)x + (u^2-v^2)y + (u^3-v^3)t + c)}$

Furthermore, if we set $v=-u$ so that f-n is independent of y, we get

$$u(x,t) = \frac{2u^2}{\cosh^2(ux + u^3t + c/2)}$$

Finally, setting $c \mapsto 0$ and using decaying at infinity, we see that

$$\boxed{u(x,t) = \frac{2u^2}{\cosh^2(ux+u^3t)} \text{ is a famous "soliton" solution of KdV eq-n: } u_t = \frac{3}{2}u\cdot u_x + \frac{1}{4}u_{xxx}}$$

- It remains to prove Theorem 2. To this end, Kac-Raina argue that "$\Gamma(u,v)^2=0$" and so "$1+a\Gamma(u,v) = \exp(a\Gamma(u,v))$", hence the result follows from Lemma 1. This is almost correct, but we will try to be more precise. We start with

$$\boxed{\begin{aligned} &\text{Lemma 2: } \Gamma(u)\Gamma(v) = :\Gamma(u)\Gamma(v):\cdot\frac{u-v}{u}, \quad \Gamma^*(u)\Gamma^*(v) = :\Gamma^*(u)\Gamma^*(v):\cdot\frac{u-v}{u} \\ &\Gamma(u)\Gamma^*(v) = :\Gamma(u)\Gamma^*(v):\cdot\frac{u}{u-v}, \quad \Gamma^*(u)\Gamma(v) = :\Gamma^*(u)\Gamma(v):\cdot\frac{u}{u-v}\end{aligned}}$$

▷ Follows from $\exp\left(-\sum_{j>0}\frac{1}{j}\left(\frac{v}{u}\right)^j\right) = \exp\left(\ln\left(1-\frac{v}{u}\right)\right) = 1-\frac{v}{u} = \frac{u-v}{u}$

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Let's use $\Gamma_+(u)$ for $\Gamma(u)$, $\Gamma_-(u)$ for $\Gamma^*(u)$. Then Lemma 2 gives:

$$\boxed{\Gamma_{\varepsilon_1}(u_1)\ldots\Gamma_{\varepsilon_n}(u_n) = :\Gamma_{\varepsilon_1}(u_1)\ldots\Gamma_{\varepsilon_n}(u_n): \cdot \prod_{i<j}\left(\frac{u_i-u_j}{u_j}\right)^{\varepsilon_i\varepsilon_j}}$$

While $:\Gamma_{\varepsilon_1}(u_1)\ldots\Gamma_{\varepsilon_n}(u_n):$ is an infinite series, we note that its matrix coeff-s on $B^{(0)}$ are all Laurent polynomials! Hence:

Corollary 3: All matrix coeff-s of $\Gamma_{\varepsilon_1}(u_1)\ldots\Gamma_{\varepsilon_n}(u_n)$ on $B^{(0)}$ are series that converge to rational f-s $\prod_{i<j}\left(\frac{u_i-u_j}{u_i}\right)^{\varepsilon_i\varepsilon_j}\cdot P$, $P\in\mathbb{C}[u_1^{\pm1},\ldots,u_n^{\pm1}]$

Now evoking that $\Gamma(u,v) = :\Gamma(u)\Gamma^*(v):$, one easily derives using Lemma 2:

$$\boxed{\Gamma(u',v')\Gamma(u,v) = :\Gamma(u,v)\Gamma(u',v'): \cdot \frac{(u'-u)(v'-v)}{(u'-v)(v'-u)}}$$

Corollary 4: $\lim\limits_{\substack{u'\to u\\ v'\to v}} \Gamma(u',v')\Gamma(u,v) = 0$ (in the sense of matrix coeff-s)

Proof of Theorem 2

- Know: $S(\tau\otimes\tau)=0$. Want: $S((1+a\Gamma(u,v))\tau\otimes(1+a\Gamma(u,v))\tau)=0$.
- By Lemma 2: $S(\Gamma(u,v)\tau\otimes\tau + \tau\otimes\Gamma(u,v)\tau) = (\Gamma(u,v)\otimes 1 + 1\otimes\Gamma(u,v))S(\tau\otimes\tau)=0$
- Finally: $S(\Gamma(u,v)\tau\otimes\Gamma(u,v)\tau) =$

$$= \lim_{\substack{u'\to u\\ v'\to v}} \frac{1}{2} S\big((\Gamma(u',v')\otimes 1 + 1\otimes\Gamma(u',v'))(\Gamma(u,v)\otimes 1 + 1\otimes\Gamma(u,v))(\tau\otimes\tau)\big)$$
$$-\frac{1}{2} S\big((\Gamma(u',v')\Gamma(u,v)\otimes 1 + 1\otimes\Gamma(u',v')\Gamma(u,v))(\tau\otimes\tau)\big) = 0$$ by Lemma 1 & Corollary 4 ∎

- We ended the class with stating the "determinant formula" for contravariant form on Verma M_λ^+ over Vir, which we'll prove next time (will leave this to the next notes).