Lecture #15

Today: Return back to the discussion of Virasoro modules, addressing:

1) when M_λ is irreducible
2) when L_λ is unitary.

Recall: $\lambda=(c,h)\in\mathbb{C}^2 \rightsquigarrow M^+_\lambda = M_\lambda = M_{c,h}$ - Verma module over Vir with h.wt. vector v_λ.

As we saw in Lecture 6: M^+_λ carries a unique contravariant form

$$\boxed{(\ ,\ ): M^+_\lambda \times M^+_\lambda \longrightarrow \mathbb{C} \ \text{ s.t. } (v_\lambda, v_\lambda)=1,\quad (L_n v_1, v_2)=(v_1, L_{-n}v_2)\ \ \forall n\ \forall v_1,v_2\in M_\lambda}$$

Moreover, different $\mathbb{Z}$-graded components of M^+_λ are orthogonal to each other. As $M^+_\lambda \simeq U(n_-)$ as $\mathbb{Z}$-graded vector spaces (by PBW theorem), if $\{x_I\}$ - basis of $U(n_-)[-n]$ for a fixed $n\in\mathbb{Z}_{>0}$, then $\{x_I(v_\lambda)\}$ - basis of $M^+_\lambda[-n]$.

Thus, one can compute $\det_n(\lambda)=\det_n(c,h)=\det\big((x_I(v_\lambda), x_J(v_\lambda))\big)$, and

$$\boxed{(\ ,\ )\text{-nondegenerate} \iff \det_n(c,h)\neq 0\ \ \forall n}$$

On the other hand, by Lectures 4-6: $M_{c,h}$-irreducible $\iff$ $(\ ,\ )$-nondegenerate

Moreover, if $\det_n(c,h)=0$ for some n, then $M_{c,h}$ has a singular vector of weight $\geq -n$. By examples from Lecture 5:

$$\det_1(c,h)=-2h,\quad \det_2(c,h)=-4h\left((2h+1)(4h+\tfrac{c}{2})-9h\right)$$

(note that $\det_1(c,h)$ divides $\det_2(c,h)$!)

Rmk: If $(c,h)\in\mathbb{R}^2$, then the same defining properties give rise to the unique Hermitian form on $M_{c,h}$. In particular, if $M_{c,h}$-unitary, then $\det_n(c,h)>0\ \forall n$

The following result will be established later in the course:

Proposition 1: For any $r,s\geq 1$, set $h_{r,s}(c):=\frac{1}{48}\left((13-c)(r^2+s^2)+\sqrt{(c-1)(c-25)}(r^2-s^2)-24rs-2+2c\right)$

Then $\det_{rs}(c, h_{r,s}(c))=0$

Note: swapping $r\leftrightarrow s$ provides two possible values of $\sqrt{(c-1)(c-25)}$.

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We shall use the above result to prove the main theorem for today:

Theorem 1 (Kac, Feigin-Fuchs): For any $m \geq 1$, we have:

$$\det_m(c,h) = \underbrace{K_m}_{\text{constant} \neq 0 \text{ (explicitly computed below)}} \cdot \prod_{\substack{r,s \geq 1 \\ rs \leq m}} (h - h_{r,s}(c))^{p(m-rs)}$$

$p(n) = \#$ partitions of size n
$= \#$ Young diagrams of n boxes

To prove, we start with a simple lemma:

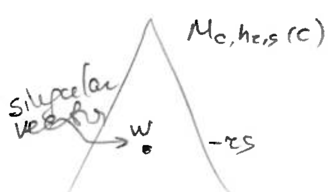
Lemma 1: If $A(t) \in Mat_{n\times n}(\mathbb{C}[t])$ with $\dim(\mathrm{Ker}(A(0))) \geq k$, then $\det A(t)$ is divisible by t^k

▸ Choose a basis $\{v_1, \ldots, v_n\}$ so that $\{v_1, \ldots, v_k\}$ – basis of $\mathrm{Ker}\, A(0)$. Then: the first k columns of $A(t)$ are divisible by t $\Rightarrow \det A(t) \vdots t^k$ ∎

Proof of Theorem

Step 1: Divisibility by $(h - h_{r,s}(c))^{p(m-rs)}$.

As we will see later Prop 1 actually says that there is a singular vector w in $M_{c,h_{r,s}(c)}$ of degree $-rs$. But by [Hwk 2, Problem 1], the



submodule generated by w is $\simeq$ Verma module, hence has dimension $p(m-rs)$ of degree $-m$ cpt

Thus, evoking Lemma 1, we see that $\det_m(c,h)$ is divisible by $(h - h_{r,s}(c))^{p(m-rs)}$

But as the numbers $\{h_{r,s}(c) \mid r,s \geq 1\}$ are pairwise distinct for generic c, we thus conclude:

$$\boxed{\det_m(c,h) \underset{\text{divisible by}}{\vdots} \prod_{\substack{r,s \geq 1 \\ rs \leq m}} (h - h_{r,s}(c))^{p(m-rs)}}$$

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(Continuation of the proof)

Step 2: Leading h-term in $\det_m(c,h)$.

Let's fix $c\in\mathbb{C}$ and consider $\det_m(c,h)$ as a polynomial of h.

$$\boxed{\begin{array}{l}\textbf{Lemma 2: } \det_m(c,h) = K_m\cdot h^{\sum_{r,s\geq 1}^{rs\leq m} p(m-rs)} + \text{lower powers of } h,\\ \text{with } K_m = \prod_{r,s\geq1}^{rs\leq m}\left((2r)^s\cdot s!\right)^{p(m-rs)}\end{array}}$$

Let's present the proof of the leading h-power, using the fact (left to hwk 4) that it comes only from the diagonal of the matrix $(x_I(V_\lambda), x_I(V_\lambda))$. The explicit calculation of K_m is also left as hwk.

▶ $(L_{-1}^{k_1}\cdots L_{-m}^{k_m}V_\lambda, L_{-1}^{k_1}\cdots L_{-m}^{k_m}V_\lambda) = (V_\lambda, L_m^{k_m}\cdots L_1^{k_1}L_{-1}^{k_1}\cdots L_{-m}^{k_m}V_\lambda)$

evoking the calculation procedure (see Lectures 4-6), we see that the above has max h-power equal to $k_1+k_2+\_+k_m$.

So: Taking product over all diagonal terms, we get:

$\sum\limits_{\substack{\mu\vdash m\\ \text{size } m \text{ partition}}}(k_1(\mu)+\_+k_m(\mu))$ and it remains to show this is $=\sum\limits_{\substack{r,s\geq1\\ rs\leq m}} p(m-rs)$

But: $p(m-rs)-p(m-r(s+1)) =: m(r,s)$ equals the number of partitions $\mu\vdash m$ that have exactly ~~r~~ repeating s times

Then: $\sum\limits_{\mu\vdash m}\sum\limits_i k_i(\mu) = \sum\limits_{\substack{r,s\geq1\\ rs\leq m}} s\,m(r,s) = \sum\limits_r\left(\sum\limits_s\left(p(m-rs)-p(m-r(s+1))\right)s\right)$

$= \sum\limits_{r,s} p(m-rs)(s-(s-1)) = \sum p(m-rs)$

▥

Step 3: Combining Steps 1+2, we immediately see that Thm 1 holds with K_m as in Lemma 2 (which is independent of c!)

▥

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Note that $h_{r,r}(c) = -\frac{(r^2-1)(c-1)}{24}$, while direct calculation shows that

$$(h - h_{r,s}(c))(h - h_{s,r}(c)) = \left(h - \frac{(r-s)^2}{4}\right)^2 + \frac{h(c-1)(r^2+s^2-2)}{24} + \frac{(r^2-1)(s^2-1)(c-1)^2}{576} + \frac{(c-1)(r-s)^2(rs+1)}{48}.$$

Corollary 1: $M_{c,h}$ is irreducible if (c,h) is not on above lines $h + \frac{(r^2-1)(c-1)}{24} = 0$ (with $r \in \mathbb{Z}_{>0}$) and above quadrics (with $r \neq s \in \mathbb{Z}_{>0}$)

In particular, investigating above eq-s, we immediately get:

Corollary 2: If $h \in \mathbb{R}_{>0}$, $c \in \mathbb{R}_{>1}$, then $M_{c,h}$ is irreducible

Finally, we also have:

Corollary 3: If $h \in \mathbb{R}_{\geq 0}$, $c \in \mathbb{R}_{\geq 1}$, then $L_{c,h}$ is unitary

▸ 1) If $h>0$, $c>1$, then $L_{c,h} \simeq M_{c,h}$. We know $M_{c',h'}$ is unitary for $c' \geq 1$, $h' > \frac{c'-1}{24}$. But since we are looking at square matrix, whose entries are pol-s of c,h, then it cannot change from positive definite to non-pos. def without passing through degenerate, which would contradict irreducibility of Verma $M(c>1, h>0)$

2) At the boundary $h=0$ or $c=1$, same argument once we pass through irreducible quotient $L_{c,h}$

∎

Rmk: In the remaining region $0<c<1$, the unitarity occurs only at the discrete series:

$$\{(c(m), h_{r,s}(m)) \mid 1 \leq s \leq r \leq m+1\} \text{ with } c(m) = 1 - \frac{6}{(m+2)(m+3)}$$

$$h_{r,s}(m) = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}$$

We will prove unitarity of $L_{c(m), h_{r,s}(m)}$ later in the course!