Lecture #16

<u>Last time</u>: Determinant formula for contravariant form on Verma / Vir
$\Downarrow$ • $M_{c,h}$ - irreducible if $c\in\mathbb{R}_{>1}$, $h\in\mathbb{R}_{>0}$
• $L_{c,h}$ - unitary if $c\in\mathbb{R}_{\geq 1}$, $h\in\mathbb{R}_{\geq 0}$.

Before we start a new topic, let's state one more corollary:

Lemma 1: a) $M_{1,h}$ is irreducible iff $h\neq \frac{m^2}{4}$ $\forall m\in\mathbb{Z}$
b) $M_{0,h}$ is irreducible iff $h\neq\frac{m^2-1}{24}$ $\forall m\in\mathbb{Z}$
c) $L_{0,h}$ is unitary iff $h=0$

► a) $h_{r,s}(1)=\frac{(r-s)^2}{24}$, with $r,s\in\mathbb{Z}_{\geq 1}$. Hence, follows from Thm 1 of Lecture 15.

b) $h_{r,s}(0)=\frac{13(r^2+s^2)+5(r^2-s^2)-24rs-2}{48}=\frac{(3r-2s)^2-1}{24}$

As r,s vary through $\mathbb{Z}_{\geq 1}$, $(3r-2s)$ varies through $\mathbb{Z}$, hence the claim.

c) Restriction of a positive definite form to any subspace is positive def!
Consider span$\{L_{-2N}v_\lambda, L_{-N}^2 v_\lambda\}$ - 2dim subspace of deg $-2N$ elements.

• $(L_{-2N}v_\lambda, L_{-2N}v_\lambda)=(v_\lambda, L_{2N}L_{-2N}v_\lambda)=(v_\lambda, (4NL_0+\frac{(2N)^3-2N}{12}C)v_\lambda)=4Nh$ (as $c=0$)

• $(L_{-2N}v_\lambda, L_{-N}^2v_\lambda)=(v_\lambda, L_{2N}L_{-N}L_{-N}v_\lambda)=(v_\lambda, L_{-N}L_{2N}L_{-N}v_\lambda)+(v_\lambda, 3NL_NL_{-N}v_\lambda)$
$=3N\cdot(v_\lambda,(2NL_0+\frac{N^3-N}{12}C)v_\lambda)=6N^2h$ (as $c=0$)

• $(L_{-N}^2v_\lambda, L_{-N}^2v_\lambda)=(v_\lambda, L_NL_NL_{-N}L_{-N}v_\lambda)=(v_\lambda, L_N(L_{-N}L_N+2NL_0+\frac{N^3-N}{12}C)L_{-N}v_\lambda)$
$=(v_\lambda, L_NL_{-N}(L_{-N}L_N+2NL_0+\frac{N^3-N}{12}C)v_\lambda)+2N\cdot(v_\lambda, L_N(L_{-N}L_0+NL_{-N})v_\lambda)$
$=2Nh\cdot(v_\lambda, L_NL_{-N}v_\lambda)+2N^2(v_\lambda, L_NL_{-N}v_\lambda)+2Nh(v_\lambda, L_NL_{-N}v_\lambda)=8N^2h^2+4N^3h$

<u>Thus</u>: $0\leq\det\begin{pmatrix}(L_{-2N}v_\lambda, L_{-2N}v_\lambda) & (L_{-2N}v_\lambda, L_{-N}^2v_\lambda)\\ (L_{-N}^2v_\lambda, L_{-2N}v_\lambda) & (L_{-N}^2v_\lambda, L_{-N}^2v_\lambda)\end{pmatrix}=4Nh(8N^2h^2+4N^3h)-6N^2h\cdot 6N^2h$
$=32N^3h^3-20N^4h^2=4N^3h^2(8h-5N)$

But if $h\neq 0$, then right-hand side <0 for $N\gg 1$ $\Rightarrow$ ↯

- For the rest of today, we shall discuss A-type affine Lie algebras. We start with $Lgl_n = gl_n[t,t^{-1}]$ which can be viewed as $Mat_{n\times n}(\mathbb{C}[t,t^{-1}])$

Thus: $Lgl_n = \{\sum_{k\in\mathbb{Z}} a_k t^k \mid a_k \in gl_n$ – all but finitely many are zero$\}$

Down-to-earth: Basis $\{\underbrace{t^k E_{ij}}_{E_{ij}(k)} \mid 1\le i,j\le n,\ k\in\mathbb{Z}\}$ and Lie bracket is

$$[E_{ij}(k), E_{i'j'}(k')] = \delta_{i'j}\, E_{ij'}(k+k') - \delta_{j'i}\, E_{i'j}(k+k')$$

Also: While $gl_n \curvearrowright \mathbb{C}^n$ with basis $e_1,\dots,e_n$, we obviously get

$$Lgl_n \curvearrowright \mathbb{C}^n[t,t^{-1}] \quad \text{via} \quad E_{ij}(k)(e_{j'}t^{k'}) = \delta_{jj'}\cdot e_i t^{k+k'}$$

Note: We can identify $\mathbb{C}^n[t,t^{-1}]$ with the same v.space V used over the last 3 weeks via $\boxed{e_i t^k \longleftrightarrow v_{i-kn}}$ ← bijection of bases. basis $\{v_i \mid i\in\mathbb{Z}\}$.

Then, the above action $Lgl_n \curvearrowright V$ can be written as follows

$$\boxed{E_{ij}(k)(v_{j'+nk'}) = \delta_{jj'}\cdot v_{i+n(k'-k)}}$$

Upshot: This immediately yields a natural embedding

$$Lgl_n \xhookrightarrow{\tau} \bar{a}_\infty \quad \text{explicitly via} \quad \sum a_k t^k \longmapsto \begin{pmatrix} \ddots & \ddots & & \\ \ddots & a_0 & a_1 & \\ & a_{-1} & a_0 & a_1 \\ & & a_{-1} & a_0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

where all blocks are $n\times n$, and one of a_0-blocks is at rows & columns $\#1,2,\dots,n$.

Down-to-earth: $E_{ij}(k) \xmapsto{\tau} \sum_{m\in\mathbb{Z}} E_{n(m-k)+i,\, nm+j}$ (which precisely follows from action)

Note: 1) τ is compatible with products, hence, with Lie brackets (so τ - Lie alg. embedding)

2) $\tau\begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ & & \ddots & 1 \\ t & 0 & \dots & 0 \end{pmatrix} = T = \sum_{i\in\mathbb{Z}} E_{i,i+1} \underset{1)}{\Longrightarrow} \tau\left(\begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ t & & & \end{pmatrix}^k\right) = T^k \quad \forall k\in\mathbb{Z}$

- Recall now that we used the Japanese 2-cocycle α in Lecture 10 to build a central extension $a_\infty = \bar{a}_\infty \oplus \mathbb{C}K$. Thus, restricting α to Lgl_n will give us an embedding of some central extension of Lgl_n into a_∞, and the next result shows this is precisely $\hat{gl}_n$ from Lecture 2:

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Lemma 2: For any $a(t)=\sum a_k t^k$, $b(t)=\sum b_k t^k \in L\mathfrak{gl}_n$, we have
$$\alpha(a(t),b(t)) = \mathrm{Res}\,(a'(t),b(t))dt = \sum_{k\in\mathbb{Z}} k\,\mathrm{Tr}(a_k b_{-k}),$$
where we use the invariant form $(\ ,\ ):\mathfrak{gl}_n\times\mathfrak{gl}_n\to\mathbb{C}$
$$(A,B)\longmapsto \mathrm{Tr}(AB)$$

Exercise: Check this by direct calculation

As a result, the affine Lie algebra $\widehat{\mathfrak{gl}}_n := L\mathfrak{gl}_n\oplus\mathbb{C}K$, defined in Lecture 2, is embedded into $\mathfrak{a}_\infty$: $\boxed{\widehat{\mathfrak{gl}}_n \hookrightarrow \mathfrak{a}_\infty}$

Likewise: If we start from $\mathfrak{sl}_n\subseteq\mathfrak{gl}_n$, then we get $L\mathfrak{sl}_n\hookrightarrow\overline{\mathfrak{a}}_\infty$, $\widehat{\mathfrak{sl}}_n\hookrightarrow\mathfrak{a}_\infty$.

In particular, evoking $\mathfrak{a}_\infty \curvearrowright \mathcal{F}^{(m)}=\Lambda^{\frac{\infty}{2},m}V$ from [Lecture 10, Thm 1], we get:

Corollary 1: The space $\mathcal{F}^{(m)}$ has a natural action of $\widehat{\mathfrak{gl}}_n$ and $\widehat{\mathfrak{sl}}_n$ (with $K\mapsto \mathrm{Id}$)

- We shall now extend the algebras $\widehat{\mathfrak{gl}}_n$ and $\widehat{\mathfrak{sl}}_n$ in order to have nondegenerate invariant pairing. To this end, we first note that the trace-form $(A,B)=\mathrm{tr}(AB)$ on $\mathfrak{gl}_n$ & $\mathfrak{sl}_n$ naturally extends to invariant nondeg. pairing on $L\mathfrak{gl}_n$ & $L\mathfrak{sl}_n$ via $(a(t),b(t))=\mathrm{Res}\,\mathrm{Tr}(a(t)b(t))dt$.

However, if we try to extend it to $\widehat{\mathfrak{sl}}_n$, then we get
$$(\underbrace{[K,a(t)]}_{0},b(t))=(K,[a(t),b(t)]) \Rightarrow (K,\underbrace{[\widehat{\mathfrak{sl}}_n,\widehat{\mathfrak{sl}}_n]}_{=\widehat{\mathfrak{sl}}_n})=0$$
Moreover, this pairing is also invariant but is clearly degenerate!

Likewise, defining the pairing on $\widehat{\mathfrak{gl}}_n$ as an extension of that on $L\mathfrak{gl}_n$ via $(K,-)=0$, we get a degenerate inv. pairing.

Let's consider the following derivation of $\widehat{\mathfrak{gl}}_n$ and $\widehat{\mathfrak{sl}}_n$:
$$\boxed{d:\ K\mapsto 0,\quad a(t)\longmapsto ta'(t)\quad \forall a(t) \text{ in } L\mathfrak{gl}_n \text{ or } L\mathfrak{sl}_n}.$$

This allows to introduce
$$\boxed{\widetilde{\mathfrak{gl}}_n := \mathbb{C}d\ltimes\widehat{\mathfrak{gl}}_n,\quad \widetilde{\mathfrak{sl}}_n=\mathbb{C}d\ltimes\widehat{\mathfrak{sl}}_n}$$

Lemma 3: The pairing on $\widehat{\mathfrak{gl}}_n \supset \widehat{\mathfrak{sl}}_n$ can be extended to a symmetric invariant non-degenerate pairing on $\widetilde{\mathfrak{gl}}_n \supset \widetilde{\mathfrak{sl}}_n$ via
$$(d,d)=0, \quad (d,K)=(K,d)=1, \quad (d,a(t))=(a(t),d)=0$$

▸ As d never arises when commuting two el-s, we need to make 2 checks:

1) $$\underbrace{([Xt^k,d],\, Yt^l)}_{=\,(-kXt^k,\, Yt^l)\,=\,-k\delta_{k,-l}\,\mathrm{tr}(XY)} \overset{?}{=} \underbrace{(Xt^k,\, [d, Yt^l])}_{=\,(Xt^k,\, l\,Yt^l)\,=\,l\,\delta_{k,-l}\,\mathrm{tr}(XY)} \quad \checkmark$$

2) $$\underbrace{([d, Xt^k],\, Yt^l)}_{=\,(kXt^k,\, Yt^l)\,=\,k\delta_{k,-l}\,\mathrm{tr}(XY)} \overset{?}{=} \underbrace{(d,\, [Xt^k, Yt^l])}_{=\,(d,\, [X,Y]t^{k+l} + k\delta_{k,-l}\,\mathrm{tr}(XY)\cdot K)} \quad \checkmark$$

So: We got a nondeg. pairing on $\widetilde{\mathfrak{gl}}_n, \widetilde{\mathfrak{sl}}_n$!

- As $\mathfrak{a}_\infty$ was naturally $\mathbb{Z}$-graded, we get a $\mathbb{Z}$-grading on $\widehat{\mathfrak{gl}}_n \supset \widehat{\mathfrak{sl}}_n$.

  Exercise: Verify it exactly coincides with the principal grading from Lecture 3.

Further, setting $\deg(d)=0$, we actually get $\mathbb{Z}$-grading on $\widetilde{\mathfrak{gl}}_n \supset \widetilde{\mathfrak{sl}}_n$.

In particular:
$$\widetilde{\mathfrak{sl}}_n = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_-, \qquad
\begin{aligned}
\widetilde{\mathfrak{n}}_+ &= \mathfrak{n}_+ \oplus \bigoplus_{k>0} \mathfrak{sl}_n \cdot t^k, & \mathfrak{n}_+ &= \text{strictly upper-}\Delta \text{ in } \mathfrak{sl}_n\\
\widetilde{\mathfrak{n}}_- &= \mathfrak{n}_- \oplus \bigoplus_{k<0} \mathfrak{sl}_n \cdot t^k, & \mathfrak{n}_- &= \text{strictly lower-}\Delta \text{ in } \mathfrak{sl}_n\\
\widetilde{\mathfrak{h}} &= \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, & \mathfrak{h} &= \text{diagonal in } \mathfrak{sl}_n
\end{aligned}$$

We also have a similar triangular decomposition for $\widetilde{\mathfrak{gl}}_n$; when needed we shall use subscript $\mathfrak{sl}_n$ or $\mathfrak{gl}_n$, like $\widetilde{\mathfrak{h}}_{\mathfrak{sl}_n} \subseteq \widetilde{\mathfrak{h}}_{\mathfrak{gl}_n}$.

Note: $\widetilde{\mathfrak{h}}_{\mathfrak{gl}_n}$ has a basis $\{E_{ii}\}_{i=1}^{n} \cup \{K, d\}$

$\widetilde{\mathfrak{h}}_{\mathfrak{sl}_n}$ has a basis $\{h_i = E_{ii} - E_{i+1,i+1}\}_{i=1}^{n-1} \cup \{h_0 = K - E_{11} + E_{nn}\} \cup \{d\}$

this specific choice will be explained later

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Def 1: a) Define $\{\widetilde{\omega}_i\}_{i=0}^{n-1} \subset (\mathfrak{h}_{\widehat{sl}_n})^*$ via $\widetilde{\omega}_i(h_j)=\delta_{ij}$, $\widetilde{\omega}_i(d)=0$.

b) Define $\{\widetilde{\omega}_m \mid m\in\mathbb{Z}\} \subset (\mathfrak{h}_{\widehat{gl}_n})^*$ via

$$\widetilde{\omega}_m(K)=1,\ \widetilde{\omega}_m(d)=0,\ \widetilde{\omega}_m(E_{ii})=\begin{cases}1, & \text{if } 1\le i\le \bar{m}\\ 0, & \text{if } \bar{m}<i\le n\end{cases} + \frac{m-\bar{m}}{n}$$

where $\bar{m}:= m \bmod n \in\{0,1,\dots,n-1\}$

Note: 1) $\widetilde{\omega}_{m+n}-\widetilde{\omega}_m \in (\mathfrak{h}_{\widehat{gl}_n})^*$ is s.t. $K\mapsto 0$, $d\mapsto 0$, $E_{ii}\mapsto 1$ $\forall i$.

2) Evoking $h_i=E_{ii}-E_{i+1,i+1}$, $h_0=K-E_{11}+E_{nn}$, we see that restricting $\widetilde{\omega}_m\in(\mathfrak{h}_{\widehat{gl}_n})^*$ under $(\mathfrak{h}_{\widehat{gl}_n})^* \twoheadrightarrow (\mathfrak{h}_{\widehat{sl}_n})^*$, we get $\widetilde{\omega}_{\bar{m}}$

Lemma 4: There exists a unique extension of $\widehat{gl}_n \curvearrowright \mathcal{F}^{(m)}$ to $\widehat{\widehat{gl}}_n \curvearrowright \mathcal{F}^{(m)}$ with $d(\psi_m)=0$.

Exercise (Homework problem): Prove it.

We finally conclude with:

Proposition 1: For any $m\in\mathbb{Z}$, the $\widehat{\widehat{gl}}_n$-module $\mathcal{F}^{(m)}$ is irreducible highest weight with the highest weight $\widetilde{\omega}_m\in(\mathfrak{h}_{\widehat{gl}_n})^*$

(we shall prove it next time)