Lecture #18

Today: Sugawara Construction
(this is a major generalization of Proposition 1 from Lecture 7).

Input: $(\mathfrak{g}, (\,,\,))$ where $\mathfrak{g}$ – finite dimensional Lie algebra / $\mathbb{C}$
$(\,,\,): \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ – invariant symmetric bilinear form

Back in Lecture 2, we introduced an affine Lie algebra $\hat{\mathfrak{g}}$ as a central extension of $L\mathfrak{g} = \mathfrak{g}[t,t^{-1}]$ via the 2-cocycle $\alpha(f(t), g(t)) = \mathrm{Res}_{t=0}(f'(t), g(t))dt$

Def 1: A $\hat{\mathfrak{g}}$-module M is called admissible if $\forall v \in M$ $(\mathfrak{g}t^n)(v) = 0$ $\forall n \geq N$ for some N – big enough.

In today's class we shall always assume that $\hat{\mathfrak{g}}$-module is admissible.
Furthermore, we shall also assume that the central element K from $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$ acts by constant k (called "level of module") on M:

$$K_{|M} = k \cdot \mathrm{Id}_M.$$

Def 2: A number k is called non-critical for $(\mathfrak{g}, (\,,\,))$ if $k\cdot(\,,\,) + \frac{1}{2}\mathrm{Kil}$ is non-degenerate

Recall: $\mathrm{Kil}(x,y) := \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}x \circ \mathrm{ad}y) = \mathrm{Tr}_{\mathfrak{g}}(z \mapsto [x,[y,z]])$ is the Killing form.

Completely analogously to Lemma 3 of Lecture 6, we have:

Lemma 1: The Witt algebra W acts by derivations on $\hat{\mathfrak{g}}$ via
$$f\partial_t(g(t), \alpha) = (fg', 0) \quad \forall f \in \mathbb{C}[t,t^{-1}],\ g \in \mathfrak{g}[t,t^{-1}],\ \alpha \in \mathbb{C}$$

Using this and $Vir \twoheadrightarrow W$, one can form the semidirect products $W \ltimes \hat{\mathfrak{g}}$ and $Vir \ltimes \hat{\mathfrak{g}}$. Note that if $L_n = -t^{n+1}\partial_t \in Vir$, $b_r = \underset{\mathfrak{g}}{b}\cdot t^r \in \hat{\mathfrak{g}}$, then: $[L_n, b_r] = -r\, b_{n+r}$.

The following is the main result for today: ↓

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Theorem 1: Let M be an admissible $\hat{\mathfrak{g}}$-module of level $k\in\mathbb{C}$, which is non-critical. Then the action $\hat{\mathfrak{g}} \curvearrowright M$ can be extended to the action $Vir \ltimes \hat{\mathfrak{g}} \curvearrowright M$ via

$$\boxed{L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{a\in B} :a_m a_{n-m}: \qquad \forall n\in\mathbb{Z}}$$

with central charge $c = k\cdot\sum_{a\in B}(a,a)$.

Here B is a basis of $\mathfrak{g}$ orthonormal with respect to $\langle\,,\,\rangle := k(\,,\,)+\frac{1}{2}Kil$,
$a_m := a\otimes t^m \in \hat{\mathfrak{g}}$ $\forall a\in\mathfrak{g}$ $\forall m\in\mathbb{Z}$, and $:a_m a_t: = \begin{cases} a_m a_t & \text{if } m\le t \\ a_t a_m & \text{if } t>m\end{cases}$

Note: $\forall v\in M$ $\forall n\in\mathbb{Z}$, $L_n(v)$ is well-defined as $\exists N\gg 0$ s.t. $(\mathfrak{g}t^{\geq N})(v)=0$.

Remarks: 1) For $\mathfrak{g}=\mathbb{C}$, $(x,y)=xy$, $M=F_\mu$, $k=1$, this recovers $Vir\ltimes A \curvearrowright F_\mu$ of Lect 7

2) One can also write L_n using any basis B' of $\mathfrak{g}$ and its dual B^*:
$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{b\in B'} :b_m b^*_{n-m}:$

The proof of the theorem is based on the following few lemmas.

Lemma 2: For any $x\in\mathfrak{g}$, we have $\sum_{a\in B}[x,a]\otimes a + a\otimes[x,a] = 0$

As $B=\{a\}$ is an orthonormal basis of $\mathfrak{g}$ w.r.t. $\langle\cdot,\cdot\rangle$, we have $y=\sum_{a'\in B}\langle y,a'\rangle a'$ for any $y\in\mathfrak{g}$. In particular, applying this to $y=[x,a]$ and using invariance:

$$\sum_{a\in B} a\otimes[x,a] = \sum_{a,a'\in B} a\otimes\langle[x,a],a'\rangle a' = -\sum_{a,a'\in B}\langle a,[x,a']\rangle a\otimes a' = -\sum_{a'\in B}[x,a']\otimes a'$$

Applying multiplication (a map $\mathfrak{g}\otimes\mathfrak{g}\xrightarrow{m} U\mathfrak{g}$), we get:

Corollary 1: For any $x\in\mathfrak{g}$, we have $\sum_{a\in B}[x,a]a + a[x,a]=0$ in $U\mathfrak{g}$

Remark: The above Lemma 2 and Corollary 1 are commonly known as saying that Casimir tensor $\Omega_0 = \sum_{b\in B'} b\otimes b^*$ and Casimir element $Cas = \sum_{b\in B'} b\cdot b^* \in U\mathfrak{g}$ being central (here $B'=\{b\}$ is any basis of $\mathfrak{g}$, and $\{b^*\}$ – dual basis w.r.t. any invariant symmetric nondeg. form on $\mathfrak{g}$).

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Lemma 3: For any $x \in \mathfrak{g}$, have: $\sum_{a\in B} [a,[a,x]] = \sum_{a \in B} Kil(x,a)a$

► Let $\{z_1, \ldots, z_m\}$ be any basis of $\mathfrak{g}$, and $\{z_1^*, \ldots, z_m^*\}$ be the dual basis of $\mathfrak{g}^*$.

By definition: $Kil(x,a) = \sum_{j=1}^m z_j^*([x,[a,z_j]])$, so that:

$$\sum_{a\in B} Kil(x,a)\,a = \sum_{a\in B}\sum_{1\le j\le m} z_j^*([x,[a,z_j]])\cdot a \overset{\text{Lemma 2}}{=\!=} \sum_{\substack{1\le j\le m\\ a\in B}} -z_j^*([x,a])\cdot[a,z_j] =$$

$$= \sum_{a\in B} -[a, \textstyle\sum_{j=1}^m z_j^*([x,a])\cdot z_j] = -\sum_{a\in B}[a,[x,a]] = \sum_{a\in B}[a,[a,x]] \quad \blacksquare$$

Evoking that $\langle\cdot,\cdot\rangle = k(\cdot,\cdot) + \frac{1}{2}Kil$, we thus get:

Corollary 2: For any $x\in\mathfrak{g}$, have $x = k\sum_{a\in B}(x,a)a + \frac{1}{2}\sum_{a\in B}[[x,a],a]$

Now we are ready to prove Theorem 1, which is equivalent to showing:

1) $[b_r, L_n] = r\, b_{r+n} \quad \forall b\in\mathfrak{g} \quad \forall r,n\in\mathbb{Z}$

2) $[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\cdot k\sum_{a\in B}(a,a)$

Proof of 1)

► As $L_n = \frac{1}{2}\sum_{a\in B}^{m\in\mathbb{Z}} :a_m a_{n-m}: = \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N} :a_m a_{n-m}:$, we have:

$$[b_r, L_n] = \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}\left([b_r,a_m]a_{n-m} + a_m\cdot[b_r,a_{n-m}]\right) =$$

$$= \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-n/2|\le N}\Big([b,a]_{r+m}a_{n-m} + a_m\cdot[b,a]_{r+n-m} + K\cdot\delta_{m,-r}\cdot r\cdot(b,a)a_{n-m} + K\cdot\delta_{m,n+r}\cdot r\cdot(b,a)a_m\Big)$$

$$= \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-n/2|\le N}\left([b,a]_{r+m}a_{n-m} + a_m\cdot[b,a]_{r+n-m}\right) + k\cdot r\cdot\sum_{a\in B}(b,a)\cdot a_{n+r}$$

$$\overset{\text{Lemma 2}}{=\!=} \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}\left([b,a]_{r+m}a_{n-m} - [b,a]_m\cdot a_{r+n-m}\right) + k\cdot r\cdot\sum_{a\in B}(b,a)\cdot a_{n+r}$$

same-looking terms, but shifted indices.

Assume $r\ge 0$ (we leave the $r<0$ case as an Exercise!). Then:

$\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}[b,a]_{r+m}\cdot a_{n-m} = \sum_{a\in B}\sum_{|m-r-\frac{n}{2}|\le N}[b,a]_m\cdot a_{r+n-m}$, which has many common terms with $\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}[b,a]_m\cdot a_{r+n-m}$; we shall depict it on picture ↓

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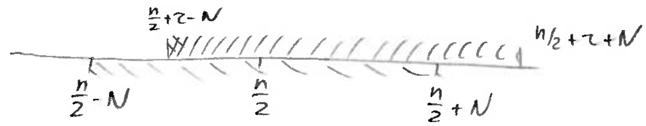

as long as $N \gg 0$.

Hence: $\frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{|m-\frac{n}{2}|\le N} \left([b,a]_{r+m}\cdot a_{n-m} - [b,a]_m \cdot a_{r+n-m}\right)$

$$= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \Bigg(\sum_{\frac{n}{2}+N<m\le \frac{n}{2}+r+N} [b,a]_m\cdot a_{r+n-m} - \underbrace{\sum_{\frac{n}{2}-N\le m<\frac{n}{2}+r-N} [b,a]_m \cdot a_{r+n-m}}_{\text{acts by } 0 \text{ on any } v\in M \text{ as } N\gg 0} \Bigg)$$

as $([b,a],a)=(b,[a,a])=0$

$$= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{\frac{n}{2}+N<m\le\frac{n}{2}+r+N} \Big(a_{r+n-m}\cdot \underbrace{[b,a]_m}_{\text{acts by } 0 \text{ on any } v\in M \text{ as } N\gg 0} + [[b,a],a]_{r+n} \Big)$$

$$= \frac{r}{2} \sum_{a\in B} [[b,a],a]_{r+n}.$$

So: $[b_r, L_n] = r\cdot\left(k\sum_{a\in B}(b,a)a + \frac{1}{2}\sum_{a\in B}[[b,a],a]\right)_{r+n} = r\cdot b_{r+n}$ by Corollary 2 ∎

Proof of 2)

- Due to 1), one gets (exactly as in Lecture 7) that:

$$[[L_n,L_m]-(n-m)L_{n+m},\ b_r]=0 \quad \forall\, b\in\mathfrak{g}\ \forall r\in\mathbb{Z} \quad \forall n,m\in\mathbb{Z}.$$

In particular, since L_0 is an infinite sum of products of a_*, we get:

$$[[L_n,L_m]-(n-m)L_{n+m},\ L_0]=0$$

But on the other hand, $[b_r, L_0]=r b_r$ $\forall b\in\mathfrak{g}$ $\forall r\in\mathbb{Z}$, so that $[-, L_0]$ picks total degree. As $[L_n,L_m]$ & L_{n+m} have all terms of degree $n+m$, we get:

$$[[L_n,L_m]-(n-m)L_{n+m},\ L_0] = (n+m)\cdot\left([L_n,L_m]-(n-m)L_{n+m}\right).$$

So: If $m\neq -n$, then $[L_n,L_m]=(n-m)L_{n+m}$, as claimed.

- Finally, for $m=-n$, we have:

$$[L_n, L_{-n}] = \Big[\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N} a_m a_{n-m},\ L_{-n}\Big]$$

$$\overset{1)}{=} \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{|m-\frac{n}{2}|\le N}\left(m\cdot a_{m-n}\cdot a_{n-m} + (n-m)a_m a_{-m}\right)$$

$$= \frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Bigg(\sum_{|m+\frac{n}{2}|\le N}(m+n)a_m a_{-m} + \sum_{|m-\frac{n}{2}|\le N}(n-m)a_m a_{-m}\Bigg)$$

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(Continuation of proof of 2))

Let's depict the segments over which the above two sums are:

and their intersection is $[\frac{n}{2}-N, -\frac{n}{2}+N]$.

Hence:

$$[L_n, L_{-n}] - 2nL_0 = \frac{1}{2}\lim_{N\to\infty} \sum_{a\in B}\left(\sum_{m=-\frac{n}{2}-N}^{-\frac{n}{2}+N} (m+n) a_m a_{-m} + \sum_{m=\frac{n}{2}-N}^{\frac{n}{2}+N} (n-m) a_m a_{-m} - \sum_{m=\frac{n}{2}-N}^{-\frac{n}{2}+N} 2n{:}a_m a_{-m}{:} \right)$$

$$= \frac{1}{2}\lim_{N\to\infty} \sum_{a\in B}\left(\underbrace{\sum_{m=-\frac{n}{2}-N}^{\frac{n}{2}-N-1} (m+n) a_m a_{-m}}_{\text{acts by } 0 \text{ on any } v\in M \text{ as } N\gg 0} + \underbrace{\sum_{-\frac{n}{2}+N+1}^{\frac{n}{2}+N} (n-m) a_{-m} a_m}_{-//-} + k\cdot(a,a)\cdot\sum_{m=1}^{-\frac{n}{2}+N} m(m+n) + k\cdot(a,a)\cdot\sum_{m=1}^{\frac{n}{2}+N} m(n-m) \right)$$

$$= \frac{1}{2}\lim_{N\to\infty} k\cdot\sum_{a\in B}(a,a)\cdot \underbrace{\left(\sum_{m=1}^{-\frac{n}{2}+N} m(m+n) + \sum_{m=1}^{\frac{n}{2}+N} m(n-m)\right)}$$

this is easily seen to be independent of N, and at $N=\frac{n}{2}$ equals $\sum_{m=1}^{n} m(n-m) = \frac{n(n+1)}{2}\cdot n - \frac{n(n+1)(2n+1)}{6} = \frac{n^3-n}{6}$

$$= \frac{n^3-n}{12} k \sum_{a\in B} (a,a)$$

This completes the proof of Theorem 1!

- Note that if $\mathfrak{g}$ – abelian fin. dim. Lie algebra, then $Kil\equiv 0$, and so $(\ ,\ )$ must be a nondeg. form while $k\neq 0$. Then $\langle\ ,\ \rangle = k(\ ,\ )$, so that the charge $c = k\sum_{a\in B}(a,a) = \sum_{a\in B}\langle a,a\rangle = \dim\mathfrak{g}$. In particular, for $\dim(\mathfrak{g})=1$, $M=F_\mu$, this recovers construction of Lecture 7.
- We concluded with the outline of the other important case: $\mathfrak{g}$-simple, but we shall finish it next time (to be included into the next notes!)