Lecture #19

Last time: Sugawara construction for any f.dim. $\mathfrak{g}$ and inv. symm. pairing $(\,,\,)$ on $\mathfrak{g}$

- Let's reformulate this in the special case: $\boxed{\mathfrak{g}\text{ - simple fin. dim.}}$

As discussed in Lecture 2, the space of inv. symm. forms on $\mathfrak{g}$ is then 1-dim, hence, any inv. symm. pairing is a multiple of Killing form.

Standard choice: $(\,,\,)$ is such that $(\theta,\theta)=2$ where $\theta\in\mathfrak{h}^*$ - highest root which will be evoked next time.

Def 1: The dual Coxeter number $h^\vee$ of $\mathfrak{g}$ is $h^\vee := 1+(\theta,\rho)$ with $\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$

Proposition 1: $Kil(a,b)=2h^\vee\cdot(a,b)$

▷ Consider the Casimir element from Lecture 18: $Cas=\sum_{a\in B'}a^2\in U(\mathfrak{g})$, where B' - orthonormal basis of $\mathfrak{g}$ w.r.t. $(\,,\,)$. Know: Cas - central.

[Exercise: Verify that on any highest weight $\mathfrak{g}$-module V of h.wt λ:
$$Cas|_V=(\lambda,\lambda+2\rho)\cdot Id_V$$

Applying this to the adjoint action $\mathfrak{g}\overset{ad}{\curvearrowright}\mathfrak{g}=\overset{=\text{ irred. of h.wt }\theta}{L_\theta}$, we get

$$Tr_{\mathfrak{g}}(Cas)=(\theta,\theta+2\rho)\cdot\dim\mathfrak{g}=(\theta,\theta+2\rho)\cdot\sum_{a\in B'}(a,a)$$
$$\| $$
$$Tr_{\mathfrak{g}}\Big(\sum_{a\in B'}ad(a)\circ ad(a)\Big)=\sum_{a\in B'}Kil(a,a)$$

As Kil & $(\,,\,)$ are multiples of each other $\Rightarrow$ $Kil(a,b)=(\theta,\theta+2\rho)\cdot(a,b)$ $\forall a,b$.

Remark: $(\theta,\theta+2\rho)=2+2(\theta,\rho)=2h^\vee$ ∎

Corollary 1: For $\mathfrak{g}$ and $(\,,\,)$ as above: $k\in\mathbb{C}$ is non-critical iff $k\neq -h^\vee$

This follows from $k(\cdot,\cdot)+\frac{1}{2}Kil=(k+h^\vee)\cdot(\cdot,\cdot)$. Moreover, the general construction of Lecture 18 can now be restated as ↓

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Theorem 1 (Sugawara for simple $\mathfrak{g}$): Let M be a level $k \neq -h^\vee$ admissible $\hat{\mathfrak{g}}$-module. Then there is an action of $Vir \ltimes \hat{\mathfrak{g}}$ on M with

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}:$$

with central charge

$$c = \frac{k}{k+h^\vee} \sum_{a\in B'} (a,a) = \frac{k \dim \mathfrak{g}}{k+h^\vee}$$

In particular, recall that $[L_0, b_r] = -r b_r \Rightarrow [-L_0, -]$ behaves as d!

Corollary 2: In the above setup, $\hat{\mathfrak{g}} \curvearrowright M$ extends to $\tilde{\mathfrak{g}} := \mathbb{C}d \ltimes \hat{\mathfrak{g}} \curvearrowright M$ via $d \mapsto -L_0$

[*Exercise*: Verify that at the critical level $k=-h^\vee$, analogous operators

$$T_n = \frac{1}{2} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}: \text{ actually commute with } \hat{\mathfrak{g}}!$$

(in fact, the center of $\mathcal{U}(\hat{\mathfrak{g}}/(k+h^\vee))^{\text{completion}}$ is known to be large!)

Finally, if M was unitary as $\hat{\mathfrak{g}}$-module, then it's also unitary as Vir-module (cf Lemma 4 of Lecture 7):

[*Exercise*: In the setup of Thm 1, if M is unitary $\hat{\mathfrak{g}}$-module, then it's unitary as Vir-module

However, for $M = L_\lambda$ to be unitary as $\hat{\mathfrak{g}}$-module we must have $k \in \mathbb{Z}_{\geq 0}$ (for $\mathfrak{sl}_n$ this was proved in Proposition 3 of Lecture 17, while the general case uses the same argument actually!), and for $k=0$ we have only the trivial module L_0. So, we may assume that $k \in \mathbb{Z}_{\geq 1}$, but then $c = \frac{k \dim \mathfrak{g}}{k+h^\vee} \geq 1$ (case-by-case check!) which means we don't get any new unitary Vir-modules.

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Our next goal is to upgrade the above construction to actually get some unitary Vir-modules with $0<c<1$. This will be based on the following Goddard-Kent-Olive construction.

Setup: • $\mathfrak{g} \supset \mathfrak{p}$ - fin. dim. Lie algebras $\rightsquigarrow \hat{\mathfrak{g}}, \hat{\mathfrak{p}}$

• $(\ ,\ )$ - inv. symm. form on $\mathfrak{g}$, hence, also on $\mathfrak{p}$

• M - admissible $\hat{\mathfrak{g}}$-module of level k ($\Rightarrow$ admissible $\hat{\mathfrak{p}}$-module)

• Assume that k is non-critical for both $\mathfrak{g}$ & $\mathfrak{p}$.

Then: Get two Virasoro actions given by $\{L_i^{\mathfrak{g}}\}$, $\{L_i^{\mathfrak{p}}\}$ on M through Sugawara construction with central charges $c_{\mathfrak{g}}, c_{\mathfrak{p}}$.

Theorem 2 (G.-K.-O. '85): Set $L_i = L_i^{\mathfrak{g}} - L_i^{\mathfrak{p}}$, $c = c_{\mathfrak{g}} - c_{\mathfrak{p}}$. Then:

1) $[L_n, L_m^{\mathfrak{p}}] = 0 \quad \forall n,m \in \mathbb{Z}$

2) $\{L_n\}_{n\in\mathbb{Z}}$ define Vir $\curvearrowright M$ of central charge c

▸ 1) $\forall b \in \mathfrak{p}, r \in \mathbb{Z}$: $[L_n^{\mathfrak{g}}, b_r] = -r\, b_{r+n} = [L_n^{\mathfrak{p}}, b_r] \Rightarrow [L_n, b_r] = 0$

$\Rightarrow [L_n, L_m^{\mathfrak{p}}] = 0$

2) $[L_n, L_m] \overset{1)}{=} [L_n, L_m^{\mathfrak{g}}] \overset{1)}{=} [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{\mathfrak{p}}] = (n-m) L_{n+m} + \delta_{n,-m} \frac{n^3-n}{12} c$ ∎

We shall now apply the above construction in very special case:

• $\mathfrak{a}$ - simple Lie algebra with inv. form $(\ ,\ )$ as on p. 1.

• $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a} \overset{\text{diag}}{\hookleftarrow} \mathfrak{a} = \mathfrak{p}$

• V', V'' - admissible $\hat{\mathfrak{a}}$-modules of levels k', k'' $\rightsquigarrow V' \otimes V''$ - admissible $\hat{\mathfrak{g}}$-module, level $= k'+k''$

• Assume: $k', k'', k'+k''$ - all non-critical.

First, applying Sugawara construction to $\hat{\mathfrak{a}}$-modules V', V'', we get

$Vir \curvearrowright V'$ with generators acting by $\{L_n'\}$, charge $= c'_{\mathfrak{a}} = \frac{k'\dim\mathfrak{a}}{k'+h^\vee}$

$Vir \curvearrowright V''$ — " — $\{L_n''\}$, charge $= c''_{\mathfrak{a}} = \frac{k''\dim\mathfrak{a}}{k''+h^\vee}$.

Then, endowing $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{a}$ with an inv. form which is direct sum of $(\,,\,)$ on each copy of $\mathfrak{a}$. Then, we can choose orthonormal basis of $\mathfrak{g}$ as a union of such in each copy of $\mathfrak{a}$. Therefore, applying Sugawara construction to $\hat{\mathfrak{g}}$-module $V'\otimes V''$, we get generators acting via $L_n^{\mathfrak{g}} = L_n'\otimes id_{V''} + id_{V'}\otimes L_n''$ and central charge $c_{\mathfrak{g}} = c'_{\mathfrak{a}} + c''_{\mathfrak{a}} = \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee}\right)\dim\mathfrak{a}$. On the other hand, viewing $V'\otimes V''$ as $\mathcal{B}_{Vir\,\hat{\mathfrak{a}}}$-module, it has charge $c_{\mathcal{B}} = \frac{k'+k''}{k'+k''+h^\vee}\dim\mathfrak{a}$ as level $= k'+k''$.

Proposition 2: In the above setup, Thm 2 translates into $Vir \curvearrowright V'\otimes V''$ with generators acting via

$$L_n = \left(\frac{1}{2(k'+h^\vee)} - \frac{1}{2(k'+k''+h^\vee)}\right)\sum_{m\in\mathbb{Z}}\sum_{a\in B'} :a_m a_{n-m}: \otimes id_{V''}$$

$$+ \left(\frac{1}{2(k''+h^\vee)} - \frac{1}{2(k'+k''+h^\vee)}\right)\sum_{m\in\mathbb{Z}}\sum_{a\in B'} id_{V'}\otimes :a_m a_{n-m}:$$

$$- \frac{1}{k'+k''+h^\vee}\sum_{m\in\mathbb{Z}}\sum_{a\in B'} a_m\otimes a_{n-m}$$

of central charge

$$c = \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee} - \frac{k'+k''}{k'+k''+h^\vee}\right)\dim\mathfrak{a}$$

In particular, if $\mathfrak{g} = \mathfrak{sl}_2$ ($\Rightarrow h^\vee = 2$), $k'=1$, $k''=m\in\mathbb{Z}_{\geq 1}$, we get

$$\boxed{c = 3\left(\frac{1}{3} + \frac{m}{m+2} - \frac{m+1}{m+3}\right) = 1 - \frac{6}{(m+2)(m+3)} = c(m)}$$ from the end of Lecture 15.

We shall next develop theory of affine Lie algebras to eventually deduce these discrete series of unitarity of Vir-modules L_λ with $c<1$