Lecture #20

Today: Contragredient Lie algebras

- But first we shall recall the basics on simple fin. dim. Lie alg-s.

Recall that $\mathfrak{g}$ is simple if $[\,,\,]\neq 0$ and $\mathfrak{g}$ has no proper ideals. The key property of simple $\mathfrak{g}$ is the root space decomposition:

$$\boxed{\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta\subset\mathfrak{h}^*\setminus\{0\}}\mathfrak{g}_\alpha}$$

$\mathfrak{h}$ = "Cartan subalgebra" ← not unique, but any two are conjugate.

such that:

1) $\mathfrak{h}$ - abelian, i.e. $[h,h']=0$ $\forall h,h'\in\mathfrak{h}$.
2) $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}\mid [h,x]=\alpha(h)\cdot x\ \ \forall h\in\mathfrak{h}\}$
3) $\mathfrak{g}_0=\mathfrak{h}$ (hence, we exclude 0 from Δ)

Note:
- $\dim(\mathfrak{g}_\alpha)=1$ $\forall\alpha\in\Delta$ $\rightsquigarrow$ can pick nonzero $e_\alpha\in\mathfrak{g}_\alpha$ $\forall\alpha$.
- Δ is a finite subset of $\mathfrak{h}^*\setminus\{0\}$ which can be described purely by combinatorial conditions
- if $\alpha,\beta\in\Delta$, then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=\begin{cases}0 & \text{if } \alpha+\beta\notin\Delta\cup\{0\}\\ \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha+\beta\in\Delta\\ \neq 0\in\mathfrak{h} & \text{if } \alpha+\beta=0\end{cases}$
- $\exists\,\bar{h}\in\mathfrak{h}$ s.t. $\alpha(\bar{h})\in\mathbb{R}\setminus\{0\}$ $\forall\alpha\in\Delta$ $\rightsquigarrow$ "polarization" $\Delta=\Delta_+\amalg\Delta_-$ where $\Delta_\pm=\{\alpha\in\Delta\mid \pm\alpha(\bar{h})>0\}$, $\Delta_-=-\Delta_+$.
- $\exists$ "simple" roots $\{\alpha_1,\alpha_2,\dots,\alpha_{n=\dim(\mathfrak{h})}\}\subseteq\Delta_+$ s.t. $\forall\alpha\in\Delta_+$ have $\alpha=k_1\alpha_1+\dots+k_n\alpha_n$ with $k_i\in\mathbb{Z}_{\geq 0}$.

For any $1\leq i\leq n=\dim\mathfrak{h}$, pick $e_i\in\mathfrak{g}_{\alpha_i}$, $f_i\in\mathfrak{g}_{-\alpha_i}$ so that $h_i:=[e_i,f_i]\in\mathfrak{h}$ satisfies $[h_i,e_i]=2e_i$, $[h_i,f_i]=-2f_i$, i.e. get $\mathfrak{sl}_2$-triple $\mathfrak{sl}_2^{(i)}=\mathrm{span}\{e_i,f_i,h_i\}$

Proposition 1: 1) $\{h_1,\dots,h_n\}$ - basis of $\mathfrak{h}$

2) $\mathfrak{g}$ is generated by $\{e_i,h_i,f_i\}_{i=1}^n$

3) The following relations hold: $[h_i,h_j]=0$, $[h_i,e_j]=\alpha_j(h_i)\,e_j$, $[h_i,f_j]=-\alpha_j(h_i)f_j$

$$[e_i,f_j]=\delta_{ij}\,h_i$$

These rel-s are clear, in particular, $[e_i,f_j]=0$ as $\alpha_i-\alpha_j\notin\Delta$ $\forall i\neq j$.

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However, the above relations do not form a set of defining rel-s of $\mathfrak{g}$. To this end, let $\tilde{\mathfrak{g}}$ be the Lie algebra generated by $3n$ el-s $\{e_i, h_i, f_i\}_{i=1}^n$ subject to the relations in part 3) of Prop 1. Then: $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$.

Unless $\mathfrak{g} \simeq \mathfrak{sl}_2$, this map has a large kernel. To describe the latter, we shall introduce the Cartan matrix $A = (a_{ij})_{i,j=1}^n$ via $a_{ij} = \alpha_j(h_i)$

[Note: It's known that $Kil|_{\mathfrak{h}\times\mathfrak{h}}$ – nondegenerate $\Rightarrow$ can identify $\mathfrak{h} \simeq \mathfrak{h}^*$.
Then $\mathfrak{h} \ni h_i \longmapsto \alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)}\alpha_i \in \mathfrak{h}^*$ and so $a_{ij} = \alpha_j(h_i) = (\alpha_j, \alpha_i^\vee) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$

Proposition 2: 1) $\mathrm{Ker}(\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g})$ contains elements

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) \quad , \quad \mathrm{ad}(f_i)^{1-a_{ij}} f_j \quad \forall i \neq j.$$

2) $\mathfrak{g}$ is generated by $\{e_i, h_i, f_i\}_{i=1}^n$ subject to rel-s

"Chevalley rel-s" $\rightsquigarrow$ $[h_i, h_j] = 0$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$, $[e_i, f_j] = \delta_{ij} h_i$ $\forall i,j$

"Serre rel-s" $\rightsquigarrow$ $[\underbrace{e_i, \ldots, [e_i}_{1-a_{ij}}, e_j]\ldots] = 0 = [\underbrace{f_i, \ldots, [f_i}_{1-a_{ij}}, f_j]\ldots]$ $\forall i \neq j$

[Example ($\mathfrak{g} = \mathfrak{sl}_n$): $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$, $h_i = E_{ii} - E_{i+1,i+1}$

$[e_i, e_j] = 0$ if $|i-j| \geq 2$

$[e_i, [e_i, e_{i+1}]] = [E_{i,i+1}, E_{i,i+2}] = 0$

$[e_i, [e_i, e_{i-1}]] = [E_{i,i+1}, -E_{i-1,i+1}] = 0$

The above is called Chevalley-Serre presentation of simple Lie algs.

Proof of 1)

Consider adjoint action $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$ with $\mathrm{ad}(x) = [x, -]$, and restrict to $\mathfrak{sl}_2^{(i)}$. Then $\mathrm{ad}(e_i)(f_j) = 0$ as $i \neq j$, $\mathrm{ad}(h_i)(f_j) = -a_{ij} \cdot f_j$. But then the $\mathfrak{sl}_2^{(i)}$-submod. of $\mathfrak{g}$ generated by f_j is highest weight module with h.wt vector f_j and is fin. dimensional as $\dim(\mathfrak{g}) < \infty$. From $\mathfrak{sl}_2$-theory this implies that it's $\simeq L_{-a_{ij}}$ (with $-a_{ij} \in \mathbb{Z}_{\geq 0}$) and so $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$

The other rel-n $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0$ is similar!

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The above Cartan matrix satisfies the following properties:

1) $a_{ii} = 2$
2) $a_{ij} \in \mathbb{Z}_{\leq 0}$ and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
3) A is indecomposable, i.e. cannot be conjugated by permutation to $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$
4) $\exists$ diagonal $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, $d_i \in \mathbb{R}_{>0}$, s.t. $D \cdot A$ - <u>symmetric</u> and defines <u>positive definite</u> quadratic form.

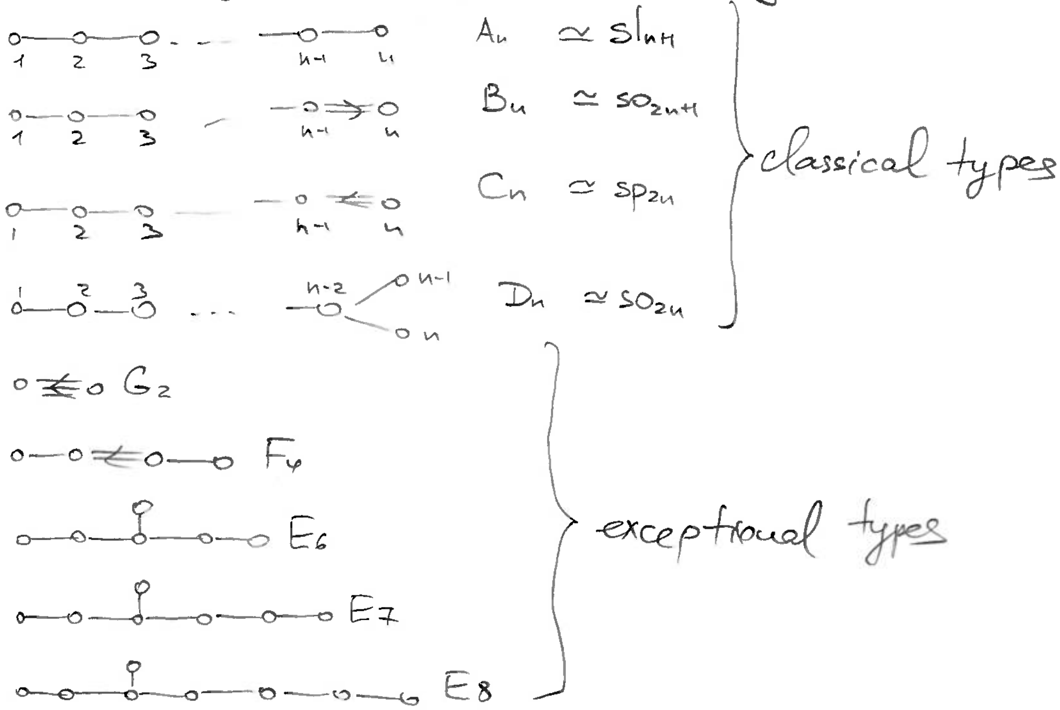
<u>Proposition 3</u>: (1) $A \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ satisfies above properties 1)-4) iff it's a Cartan matrix of a simple fin. dim. Lie algebra.

(2) A complete classification of such matrices and hence simple fin. dim. Lie algebras is given by their Dynkin diagrams. The latter are graphs with n vertices and the following edges:

- if $a_{ij} = 0 = a_{ji}$ $\Rightarrow$ no edges b/w i & j
- if $a_{ij} = -1 = a_{ji}$ $\Rightarrow$ single unoriented edge $i - j$
- if $a_{ij} = -2$, $a_{ji} = -1$ $\Rightarrow$ double edge $\underset{i}{\circ} \Leftarrow \underset{j}{\circ}$
- if $a_{ij} = -3$, $a_{ji} = -1$ $\Rightarrow$ triple edge $\underset{i}{\circ} \Lleftarrow \underset{j}{\circ}$
- no loops

<u>Note</u>: These are all possibilities as $\begin{pmatrix} 2 = a_{ii} & a_{ij} \\ a_{ji} & 2 = a_{jj} \end{pmatrix}$ is positive definite $\Rightarrow 4 - a_{ij}a_{ji} > 0$

Here is the full list of Dynkin diagrams



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- Now we are ready to introduce contragredient Lie algebras.

Input: $A \in Mat_{n\times n}(\mathbb{C})$.

Let $Q = \mathbb{Z}^{\oplus n} = \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_n$ be a free rank n abelian gp ("root lattice")

Def 1: A contragredient Lie algebra corresponding to $A \in Mat_{n\times n}(\mathbb{C})$ is a Q-graded Lie algebra over $\mathbb{C}$, denoted $\mathfrak{g}(A) = \mathfrak{g}$, generated by $\{e_i, h_i, f_i\}_{i=1}^n$ s.t.:

(1) $[h_i, h_j] = 0$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$, $[e_i, f_j] = \delta_{ij} h_i$

(2) $\mathfrak{g}_0$ has basis $\{h_1, \ldots, h_n\}$, $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$

(3) any nonzero Q-graded ideal $J \subseteq \mathfrak{g}(A)$ has nonzero intersection with $\mathfrak{g}_0$

Example: If A is a Cartan matrix of simple f. dim. $\mathfrak{g}$, then $\mathfrak{g}$ satisfies above (property (3) is obvious as there are no proper ideals at all)

Proposition 4: For any $A \in Mat_{n\times n}(\mathbb{C})$, $\exists!$ contragredient Lie alg. up to isomorphism

▶ Similarly to page 2, consider $\tilde{\mathfrak{g}}(A)$ gen-d by $\{e_i, h_i, f_i\}_{i=1}^n$ subject to the relations in part (1). We claim that (2) then naturally holds, which follows from the following homework exercise:

Exercise: $\tilde{\mathfrak{g}}(A)$ admits a triangular decomposition $\tilde{\mathfrak{g}}(A) \simeq \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ where $\tilde{\mathfrak{n}}_-$ is generated by $\{f_i\}$, $\tilde{\mathfrak{h}}$ – by $\{h_i\}$, $\tilde{\mathfrak{n}}_+$ by $\{e_i\}$.
Moreover: $\tilde{\mathfrak{h}}$ has a basis $\{h_i\}_{i=1}^n$, $\tilde{\mathfrak{n}}_+ =$ Free Lie alg. in $\{e_i\}$
$\tilde{\mathfrak{n}}_- =$ Free Lie alg. in $\{f_i\}$

Then clearly we must have $\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A)$. Moreover, as a sum of Q-graded ideals is Q-graded (and if $\underset{Q\text{-graded}}{J_1, J_2} \cap \tilde{\mathfrak{h}} = 0 \Rightarrow (J_1 + J_2) \cap \tilde{\mathfrak{h}} = 0$)
we have $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/I_{max}$, where $I_{max} = \sum\limits_{\substack{\text{graded ideals} \\ J \cap \tilde{\mathfrak{h}} = 0}} J$ – maximal Q-graded ideal $\cap \tilde{\mathfrak{h}} = 0$.

Clear: $\{h_i\}_{i=1}^n$ – basis of $\mathfrak{g}_0 = \mathfrak{g}(A)_0 = \tilde{\mathfrak{g}}(A)_0$

$f_i, e_i \notin I_{max}$ (as otherwise would have $h_i \in I_{max} \Rightarrow ⚡$) $\Rightarrow \mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$

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In general, not much is known about $\mathfrak{g}(A)$. So we shall restrict to:

Def 2: $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is called "generalized Cartan matrix" if

1) $a_{ii} = 2 \ \forall i$
2) $a_{ij} \in \mathbb{Z}_{\leq 0}$, $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \quad \forall i \neq j$
3) A is symmetrizable, i.e. $\exists D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ with $d_i \in \mathbb{R}_{>0}$ s.t. DA-symmetric

Note: A generalized Cartan matrix is the usual Cartan if DA-positive definite

Theorem 1 (Gabber–Kac 1981): For a generalized Cartan matrix A, the ideal $I_{max} = \mathrm{Ker}(\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A))$ is generated by Serre rel-s $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j)$ & $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) \quad \forall i \neq j$

Proof of easy part: rel-s do hold

► Pick any $i \neq j$ and consider element $X_{ij} = \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) \in \tilde{\mathfrak{n}}_-$. It suffices to show that $[e_k, X_{ij}] = 0 \ \forall k$, since then the ideal generated by X_{ij} belongs to $\tilde{\mathfrak{n}}_-$ $\Rightarrow$ has zero intersection with $\mathfrak{h} = \mathfrak{g}_0$ $\Rightarrow$ is zero in $\mathfrak{g}(A)$!

- If $k \neq i, j$, then $[e_k, f_i] = 0 = [e_k, f_j] \Rightarrow [e_k, X_{ij}] = 0$
- If $k = j$, then $[e_k, f_i] = 0$, $[e_k, f_j] = h_j \Rightarrow [e_k, X_{ij}] = \mathrm{ad}(f_i)^{1-a_{ij}}(h_j)$

  Now if $a_{ij} \leq -1 \Rightarrow [f_i, h_j] = a_{ji} f_i \Rightarrow [f_i, [f_i, h_j]] = 0$

  Now if $a_{ij} = 0 \Rightarrow a_{ji} = 0 \Rightarrow [f_i, h_j] = 0$.
- If $k = i$, then we again consider $\mathfrak{sl}_2^{(i)} = \mathrm{span}\{e_i, h_i, f_i\}$ and its action on f_j. However, in contrast to our proof of Prop 2 part 1), we do not have $\dim(\mathfrak{g}_-) < \infty$. Instead, using $\mathfrak{sl}_2$-representation theory again, we see that $\mathrm{ad}(f_i)^{1-a_{ij}} f_j$ is a singular vector w.r.t. $\mathfrak{sl}_2^{(i)}$ $\Rightarrow$ $\Rightarrow [e_i, X_{ij}] = 0$ ∎

Def 3: A Kac-Moody algebra is a Lie algebra $\mathfrak{g}(A)$ for generalized Cartan A.